Can Thermalization in Heavy Ion Collisions be Described by QCD Diagrams?

Yuri V. Kovchegov*

Department of Physics, The Ohio State University
Columbus, OH 43210

March, 2005

Abstract

The onset of thermalization in heavy ion collisions in the weak coupling framework can be viewed as a transition from the initial state Color Glass Condensate dynamics, characterized by the energy density scaling like $\epsilon \sim 1/\tau$ with τ the proper time, to the hydrodynamics-driven expansion of the quark-gluon plasma with $\epsilon \sim 1/\tau^{4/3}$ (or higher power of $1/\tau$ for the boost non-invariant case). We argue that, at any order of the perturbative expansion in the QCD coupling constant, the gluon field generated in an ultrarelativistic heavy ion collision leads to energy density scaling as $\epsilon \sim 1/\tau$ for late times $\tau \gg 1/Q_s$. Therefore it is likely that thermalization and hydrodynamic description of the gluon system produced in heavy ion collisions can not result from perturbative QCD diagrams at these late times. At earlier times with $\tau \sim 1/Q_s$ the subleading corrections to ϵ in $1/\tau$ expansion (terms scaling like $\sim 1/\tau^{1+\Delta}$ with $\Delta > 0$) may become important possibly leading to hydrodynamic-like behavior of the gluon system. Still, we show that such corrections do not contribute to the particle production cross section, and are likely to be irrelevant for physical observables. We generalize our results by including massless quarks into the system. Thus, it appears that the apparent thermalization of quarks and gluons, leading to success of Bjorken hydrodynamics in describing heavy ion collisions at RHIC, can only be attributed to the non-perturbative QCD effects.

*e-mail: yuri@mps.ohio-state.edu

1 Introduction

Understanding how the system of quarks and gluons produced in ultrarelativistic heavy ion collisions evolves towards thermal equilibrium is one of the central questions in our theoretical understanding of nuclear collisions. On the one hand, there exists a strong experimental evidence for equilibration of the quark-gluon system produced in a heavy ion collision at RHIC. The evidence is based on the success of hydrodynamic models of the collisions [1, 2, 3, 4, 5], indicating a collective behavior of the quark-gluon system, and on the discovery of jet quenching [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], which demonstrated the presence of strong final state interactions. However, hydrodynamic models of the evolution of the quark gluon system only work well, especially for the elliptic flow v_2 [19], if equilibration occurs in fact at a *very* early time [3, 4], $t \lesssim 0.5\,\text{fm}/c$, a time whose smallness is difficult to reconcile with current dynamical pictures of equilibration [20, 21, 22, 23, 24, 25, 26].

On the other hand, a complete theoretical understanding of thermalization is still lacking. The success of saturation/Color Glass approach [27, 28, 29, 30, 31, 32, 33, 34, 35, 36] in describing particle multiplicities [37] in heavy ion collisions and particle spectra in deuteron–gold collisions [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51] (see also [52, 53]) appears to indicate that saturation/Color Glass formalism is valid for the initial stages of heavy ion collisions at RHIC. Our understanding of the very early pre-equilibration stages of the collision and their description in terms of classical gluon fields has significantly advanced in the recent years [54, 55, 56, 57, 58, 59, 60, 61]. Based on this saturation initial conditions, Baier, Mueller, Schiff and Son proposed the so-called "bottom-up" thermalization scenario [20] in which multiple $2 \to 2$, $2 \to 3$ and $3 \to 2$ rescattering processes, the importance of which was originally underlined in [62], would drive the system to thermal equilibration over the time scales of the order of $\tau_0 \sim 1/\alpha_s^{13/5} Q_s$. While the estimates in [20] were mostly parametric, the numerical value of this thermalization time appears to be much larger than 0.5 fm needed by hydrodynamic simulations [3, 4].

More recently it was argued by Arnold, Lenaghan and Moore that the "bottom-up" thermalization scenario could be susceptible to plasma instabilities [63, 64], which were advocated previously in [65, 66, 67, 68, 69]. Such instabilities might help facilitate the equilibration process making the thermalization time shorter than predicted by the "bottom-up" scenario. However, in [64] Arnold and Lenaghan proved a lower bound on the thermalization time, which happened to be surprisingly close to the "bottom-up" estimate. In [70] it has been suggested that, while complete thermalization may not happen until later times, an isotropization of the produced particle distribution in momentum space may happen much faster, leading to generation of longitudinal pressure needed for hydrodynamic description to work.

Here we take a different approach to the problem of thermalization. Thermalization could be thought of as a transition between the initial conditions, which are characterized by the energy density scaling like $\epsilon \sim 1/\tau$, and the Bjorken hydrodynamics, which, in case of the ideal gas equation of state has $\epsilon \sim 1/\tau^{4/3}$ [1]. (Of course at realistic temperatures achieved in heavy ion collisions the power of 4/3 may become somewhat smaller: however, it is always greater than 1 for hydrodynamic expansion.) Therefore it appears that corrections to the saturation/Color Glass initial conditions [54, 55, 56, 57, 58, 59] would contribute towards modifying the $\epsilon \sim 1/\tau$ scaling to some higher power. Thus one should be interested in Feynman diagrams which would bring in τ-dependent corrections to $\epsilon \sim 1/\tau$ scaling of the (classical) gluon fields in the initial

stages of the collisions. Unfortunately, after examining a number of diagrams, we noticed that while many of them introduce τ-dependent corrections to the initial conditions, such corrections are subleading and small at large τ and do not modify $\epsilon \sim 1/\tau$ scaling at late times. After reaching this conclusion we have constructed a general argument proving that $\epsilon \sim 1/\tau$ scaling always dominates at late times, both for classical fields and quantum corrections, which we are presenting here.

The paper is structured in the following way. We begin in Section 2 by calculating the energy density of a lowest-order non-trivial classical gluon field from [55]. As expected the energy density of the classical field scales as $\epsilon \sim 1/\tau$. We then continue in Section 3 by considering the most general case of boost-invariant gluon production, which is, indeed, not limited to classical fields. We argue that $\epsilon \sim 1/\tau$ scaling persists to all orders in the coupling constant α_s, as shown in Eq. (53). The argument is based on a simple observation (see Eq. (A1)) that τ-dependent corrections to the classical gluon field may only come in through powers of gluon virtuality k^2 in momentum space with each power of k^2 giving rise to a power of $1/\tau$. In order for the on-mass shell amplitude (at $k^2 = 0$) to be non-singular only positive powers of k^2 are allowed: hence, the corrections come in only as inverse extra powers of τ and are negligible at late times. In Section 3 we generalize our results to rapidity-dependent distributions. The $\epsilon \sim 1/\tau$ scaling does not get modified by rapidity-dependent corrections either (see Eq. (83)). Rapidity-dependent corrections come in as, for example, powers of k_+, which is one of light cone components of the gluon's momentum. However, as could be seen from, say, Eq. (B7), powers of k_+ do not modify the τ-dependence of energy density. In Section 4 we argue that $\epsilon \sim 1/\tau$ scaling persists even when massless quarks are included in the problem. Therefore it appears that perturbative thermalization can not happen in heavy ion collisions. We conclude in Section 5 by arguing that if perturbative thermalization is impossible, than the non-perturbative QCD effects must be responsible for the formation of quark-gluon plasma (QGP) at RHIC [71]. We list the non-perturbative effects which we believe may be responsible for thermalization.

2 Energy-Momentum Tensor of Classical Gluon Field

We start by calculating the energy-momentum tensor of the lowest order gluon field produced in an ultrarelativistic heavy ion collision. This field has been found analytically in [54, 55] and the corresponding Feynman diagrams are depicted here in Fig. 1. The cross in Fig. 1 denotes the space-time point in which we measure the gluon field. The gluon field in $\partial_\mu A^\mu = 0$ covariant gauge given by diagrams in Fig. 1 can be written as [55]

$$A_\mu^{(3)\,a}(x) = -i \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{-ik\cdot x}}{k^2 + i\epsilon k_0} \, J_\mu^{(3)\,a}(k), \tag{1}$$

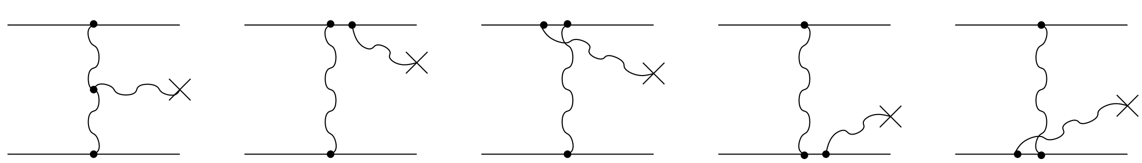

Figure 1: Lowest order ($\sim g^3$) gluon field produced in nuclear collisions.

with

$$J_\mu^{(3)\,a}(k) \; = \; \sum_{i,j=1}^{A_1,A_2} \int d^2 q \, \frac{g^3}{(2\pi)^2} \, f^{abc} \, (T_i^b) \, (\tilde{T}_j^c) \, e^{i[k_+ x_{i-} + k_- y_{j+} - \underline{k} \cdot \underline{y}_j - \underline{q} \cdot (\underline{x}_i - \underline{y}_j)]} \, \frac{C_\mu(k, \underline{q})}{\underline{q}^2 (\underline{k} - \underline{q})^2} \tag{2}$$

where $C_\mu(k, \underline{q})$ is the Lipatov vertex [73]

$$C_\mu(k, \underline{q}) \; = \; \left(\frac{\underline{q}^2}{k_- + i\epsilon} - k_+ \, , \; - \frac{(\underline{k} - \underline{q})^2}{k_+ + i\epsilon} + k_- \, , \; 2\underline{q} - \underline{k} \right). \tag{3}$$

The field of Eq. (1) is given for a collision of a quark i in one of the nuclei having transverse coordinate $\underline{x}_i$ and light cone coordinate x_{i-} with a quark j in the other nucleus having transverse coordinate $\underline{y}_j$ and the light cone coordinate y_{j+}. The matrices (T_i^b) and $(\tilde{T}_j^c)$ act in the color spaces of the quarks i and j correspondingly. Indeed we sum over all quark pairs in Eq. (2), with valence quarks i being in any of the A_1 nucleons in the first nucleus and valence quarks j being in any one of the A_2 nucleons in the second nucleus.

The energy-momentum tensor of a gluon field is given by

$$T^{\mu\nu} \; = \; -F^{a\,\mu\rho} \, F^{a\,\nu}{}_{\rho} + \frac{1}{4} \, g^{\mu\nu} \, (F^a_{\rho\sigma})^2. \tag{4}$$

We need to calculate $T^{\mu\nu}$ averaged in the wave functions of both nuclei [31, 55]

$$\langle T^{\mu\nu} \rangle \; = \; \left\langle -F^{a\,\mu\rho} \, F^{a\,\nu}{}_{\rho} + \frac{1}{4} \, g^{\mu\nu} \, (F^a_{\rho\sigma})^2 \right\rangle, \tag{5}$$

where the averaging implies integrating over all possible positions of quarks in the nucleons and nucleons in the nuclei, and taking traces (divided by N_c) in the color spaces of the quarks [31, 55]. The averaging can be represented as

$$\langle \ldots \rangle \; = \; \prod_{i,j=1}^{A_1,A_2} \frac{d^2 x_i}{S_{1\perp}} \frac{d^2 y_j}{S_{2\perp}} \frac{d x_{i-}}{a_-} \frac{d y_{j+}}{a_+} \, \frac{1}{N_c^2} \, \text{Tr}_i [\text{Tr}_j [\ldots]] \tag{6}$$

where $S_{1\perp}$ and $S_{2\perp}$ are the cross sectional areas of the two nuclei, which we for simplicity assume to be cylindrical with the cylinder axis pointing in the beam (z-) direction. a_- and a_+ are Lorentz-contracted nucleon sizes in the $-$ and $+$ directions correspondingly, which are very small, making averaging over x_{i-} and y_{j+} equivalent to just putting $x_{i-} = 0$ and $y_{j+} = 0$ [55].

We are interested in calculating $T_{\mu\nu}$ for the gluon field produced in a central nuclear collisions in the forward light cone. Since the gluon field of Eq. (1) is $o(g^3)$, we may only use it to compute $o(g^6)$ contribution to the energy-momentum tensor, for which we will need only the Abelian part of the field strength tensor $F^a_{\mu\nu}$. To compute higher orders in $T_{\mu\nu}$ one would also need higher orders in A^a_μ. Using the field (1) in the Abelian part of Eq. (5), performing the averaging defined in Eq. (6) and remembering that the multiplicity distribution given by the diagrams of Fig. 1 for gluons with transverse momentum $\underline{k}$, rapidity y, located at impact parameter $\underline{b}$, is

$$\frac{dN}{d^2 k \, dy \, d^2 b} \; = \; \frac{8 \, \alpha_s^3 \, C_F}{\pi} \, \frac{A_1 \, A_2}{S_{1\perp} \, S_{2\perp}} \, \frac{1}{\underline{k}^4} \, \ln \frac{k_T}{\Lambda} \tag{7}$$

we obtain after lengthy algebra (and dropping the averaging sign around $T_{\mu\nu}$)

$$\begin{aligned}
T_{++} &= \left(\frac{x_+}{\tau}\right)^2 \pi \int d^2k \, \frac{dN}{d^2k\, d\eta\, d^2b} \, k_T^2 \, [J_1(k_T\tau)]^2 \\
T_{--} &= \left(\frac{x_-}{\tau}\right)^2 \pi \int d^2k \, \frac{dN}{d^2k\, d\eta\, d^2b} \, k_T^2 \, [J_1(k_T\tau)]^2 \\
T_{+-} &= \frac{x_+ \, x_-}{\tau^2} \, \pi \int d^2k \, \frac{dN}{d^2k\, d\eta\, d^2b} \, k_T^2 \, [J_0(k_T\tau)]^2 \\
T_{ij} &= \delta_{ij} \, \frac{\pi}{2} \int d^2k \, \frac{dN}{d^2k\, d\eta\, d^2b} \, k_T^2 \, [J_0(k_T\tau)]^2 \, , \quad T_{+i} \, = \, T_{-i} \, = \, 0.
\end{aligned} \tag{8}$$

Here we used $x_\pm = (t \pm z)/\sqrt{2}$, $\tau = \sqrt{t^2 - z^2} = \sqrt{2 x_+ x_-}$, and $k_T = |\underline{k}|$. We also took advantage of the fact that the multiplicity distribution (7) is rapidity independent and, since $T_{\mu\nu}$ should depend only on space-time coordinates, replaced momentum space rapidity y with the space-time rapidity $\eta = (1/2)\ln(x_+/x_-)$. (At this point such substitution makes no difference: in Section 4.2 we will show how this substitution is formally justified in the rapidity-dependent case.) In arriving at Eq. (8) we have used the integral defined in Eq. (A1) and the one given by Eq. (B5) in Appendix B with $\Delta = 0$ along with

$$\int \frac{d^2q}{\underline{q}^2 (\underline{k} - \underline{q})^2} \, = \, \frac{4\pi}{\underline{k}^2} \, \ln \frac{k_T}{\Lambda} \tag{9}$$

where Λ is some infrared cutoff. Eq. (8) is derived in the leading logarithmic approximation in $\ln k_T/\Lambda$.

Eq. (8) gives us the energy-momentum tensor in the forward light cone of the lowest order gluon field from Fig. 1 produced in a central collision of two identical nuclei. While it is written in a non-specific form with regards to the order of the coupling constant g, we have proven Eq. (8) only at the order $o(g^6)$.

3 Energy Density in the Boost-Invariant Approximation

3.1 Region of Applicability

Let us first consider the case of high energy heavy ion collisions, where the total rapidity interval is large enough to allow for eikonal approximation, but not large enough for quantum BFKL-type corrections [73] to become important. This is the quasi-classical regime of McLerran-Venugopalan model [30, 31, 32]. To achieve it one needs the Bjorken x variable to be small enough such that [74]

$$x \, < \, \frac{1}{2 m_N R} \, \sim \, A^{-1/3}, \tag{10}$$

which is the condition ensuring that coherent eikonal interactions are possible in the nuclear wave functions. For corresponding rapidities, $Y = \ln 1/x$, the condition of Eq. (10) means that

$$Y \, > \, \ln A^{1/3}. \tag{11}$$

Remembering that McLerran-Venugopalan model corresponds to resummation of multiple rescatterings parameter $\alpha_s^2 A^{1/3} \sim 1$ we rewrite Eq. (11) as

$$Y \; > \; \ln \frac{1}{\alpha_s^2} \; \sim \; \ln \frac{1}{\alpha_s}. \tag{12}$$

On the other hand, the boost invariant approximation is broken down by quantum evolution corrections, which, in the dominant leading logarithmic approximation, bring in powers of $\alpha_s Y$ [73, 33, 34, 35]. Indeed these corrections are negligible when $\alpha_s Y \lesssim 1$, such that

$$Y \; < \; \frac{1}{\alpha_s}. \tag{13}$$

Eqs. (12) and (13) define the rapidity interval for nuclear collisions in which the boost invariant approximation, which we will consider in this Section, is valid.

3.2 Most General Boost Invariant form of $T^{\mu\nu}$

Similar to the Bjorken approach [1] we will consider a central collision of two very large nuclei, such that the gluon production is translationally invariant in the transverse direction. Defining two four-vectors in terms of light-cone coordinates

$$u_\mu = \left(\frac{x_+}{\tau}, \frac{x_-}{\tau}, \underline{0} \right) \tag{14}$$

and

$$v_\mu = \left(\frac{x_+}{\tau}, -\frac{x_-}{\tau}, \underline{0} \right) \tag{15}$$

we can write the most general energy-momentum tensor for the system as

$$T_{\mu\nu} \; = \; A(\tau)\, u_\mu u_\nu + B(\tau)\, (u_\mu v_\nu + u_\nu v_\mu) + C(\tau)\, v_\mu v_\nu + D(\tau)\, g_{\mu\nu}, \tag{16}$$

where in our convention $g_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$. Here the parameters A, B, C, D in Eq. (16) are functions of τ only, since the large transverse extent and azimuthal cylindrical symmetry for central collisions of the nuclei allow us to neglect the transverse coordinate dependence, and the assumption of boost invariance makes functions A, B, C, D independent of space-time rapidity $\eta = (1/2) \ln(x_+/x_-)$.

From Eq. (16) we see that

$$T_{++} \; = \; [A(\tau) + B(\tau) + C(\tau)] \left(\frac{x_+}{\tau} \right)^2 \tag{17}$$

and

$$T_{--} \; = \; [A(\tau) - B(\tau) + C(\tau)] \left(\frac{x_-}{\tau} \right)^2. \tag{18}$$

Due to $+ \leftrightarrow -$ symmetry of the collision of two identical nuclei it should be possible to obtain T_{--} from T_{++} after changing all $+$ indices to $-$ indices of all the relevant four-vectors in it. This condition, when applied to Eqs. (17) and (18), demands that $B(\tau) = 0$.

Rewriting the remaining non-zero functions A, C, D as

$$A(\tau) = \epsilon(\tau) + p(\tau), \quad C(\tau) = p_3(\tau) - p(\tau), \quad \text{and} \quad D(\tau) = -p(\tau) \tag{19}$$

we obtain for the non-zero components of the energy momentum tensor

$$\begin{aligned}
T_{++} &= [\epsilon(\tau) + p_3(\tau)] \left(\frac{x_+}{\tau}\right)^2, \\
T_{--} &= [\epsilon(\tau) + p_3(\tau)] \left(\frac{x_-}{\tau}\right)^2, \\
T_{+-} &= [\epsilon(\tau) - p_3(\tau)] \frac{x_+ x_-}{\tau^2} = [\epsilon(\tau) - p_3(\tau)] \frac{1}{2}, \\
T_{ij} &= \delta_{ij} \, p(\tau),
\end{aligned} \tag{20}$$

where the indices $i, j = 1, 2$ denote the transverse components of the tensor. Eq. (20) gives us the most general boost-invariant energy-momentum tensor for a collision of two very large nuclei with the total rapidity interval satisfying conditions (12) and (13) allowing for a boost-invariant description of the gluon production.

At $z = 0$ in the center-of-mass frame the energy-momentum tensor from Eq. (20) can be written as

$$T^{\mu\nu} = \begin{pmatrix} \epsilon(\tau) & 0 & 0 & 0 \\ 0 & p(\tau) & 0 & 0 \\ 0 & 0 & p(\tau) & 0 \\ 0 & 0 & 0 & p_3(\tau) \end{pmatrix}. \tag{21}$$

Now we can see the physical meaning of the parameterization introduced in Eq. (19): ϵ is the energy density and p_3 is the longitudinal pressure along the beam axis (z-direction), which in principle does not have to be equal to the transverse pressure p. Indeed, for the case of boost-invariant Bjorken hydrodynamics [1], the two pressures are identical, $p_3(\tau) = p(\tau)$. However, as one can show, for classical gluon fields generated in heavy ion collisions [54, 55, 56, 57, 58], the longitudinal pressure component is zero at sufficiently late times, $p_3(\tau) = 0$, while $\epsilon(\tau) = 2\, p(\tau) \neq 0$ [59].

Applying the conservation of energy-momentum tensor condition

$$\partial_\mu T^{\mu\nu} = 0 \tag{22}$$

to the tensor in Eq. (20) we obtain

$$\frac{d\epsilon}{d\tau} = -\frac{\epsilon + p_3}{\tau} \tag{23}$$

similar to Bjorken hydrodynamics [1].

Eq. (23) shows that if energy density scales with proper time as $\epsilon \sim 1/\tau$ then the longitudinal pressure is zero, $p_3 = 0$. This is indeed the case for classical gluon production in the initial stages of the heavy ion collisions considered in Section 2 (see also [59]). To see this let us use the energy momentum tensor from Eq. (8) in Eq. (20) to write

$$\epsilon = \frac{\pi}{2} \int d^2k \, \frac{dN}{d^2k \, d\eta \, d^2b} \; k_T^2 \, \left\{ [J_1(k_T \tau)]^2 + [J_0(k_T \tau)]^2 \right\}$$

$$p_3 = \frac{\pi}{2} \int d^2k \, \frac{dN}{d^2k \, d\eta \, d^2b} \, k_T^2 \, \left\{ [J_1(k_T \tau)]^2 - [J_0(k_T \tau)]^2 \right\}$$

$$p = \frac{\pi}{2} \int d^2k \, \frac{dN}{d^2k \, d\eta \, d^2b} \, k_T^2 \, [J_0(k_T \tau)]^2 \, . \tag{24}$$

Using the large-argument asymptotics of the Bessel functions we write

$$\epsilon \Big|_{\tau \gg 1/\langle k_T \rangle} \approx \frac{1}{\tau} \int d^2k \, \frac{dN}{d^2k \, d\eta \, d^2b} \, k_T = \frac{1}{\tau} \, \frac{dE_T}{d\eta \, d^2b}, \tag{25}$$

which precisely agrees with the Bjorken energy density estimate [1]. Here we assumed that the gluon spectrum is characterized by some typical transverse momentum $\langle k_T \rangle$, such that large time asymptotics is defined by $\tau \gg 1/\langle k_T \rangle$. (Strictly speaking such "typical" momentum for lowest order gluon field of Eq. (1) is the infrared cutoff Λ, but it would become the saturation scale $Q_s \gg \Lambda$ once multiple rescatterings are included [57, 58].)

Similarly, using the large-argument asymptotics of the Bessel functions one can show that

$$p_3 \Big|_{\tau \gg 1/\langle k_T \rangle} \approx 0 \tag{26}$$

in agreement with Eq. (25) and Eq. (23). Thus the large time asymptotics of the energy-momentum tensor of the lowest order classical gluon field is given by $T_{\mu\nu} = \text{diag}\{\epsilon, p, p, 0\}$ with $\epsilon = 2\,p$ and ϵ given by Eq. (25)[1]. Similar results were obtained in numerical simulations of the full classical gluon field including all orders in multiple rescatterings [59].

The onset of thermalization or isotropization of the system [70] should come with generation of the non-zero longitudinal pressure p_3 comparable to the transverse pressure p. In order for that to happen Eq. (23) necessarily requires the energy density to start scaling with τ as $\epsilon \sim 1/\tau^{1+\Delta}$, where Δ is some positive number. In the case of ideal Bjorken hydrodynamics $\Delta = 1/3$.

Thus the process of thermalization in heavy ion collisions can be viewed as a transition from the $\epsilon \sim 1/\tau$ scaling, characteristic of free-streaming classical fields (25), to $\epsilon \sim 1/\tau^{1+\Delta}$ scaling. Below we are going to study whether such transition can result from Feynman diagram resummation.

3.3 Can Boost-Invariant Bjorken Hydro Result from Feynman Diagrams?

Let us explore what kinds of energy-momentum tensor may result from Feynman diagram resummation. We will concentrate only on gluon fields, and later will generalize our conclusions to include quark fields as well. We will assume that the initial gluon field is given by the classical field of McLerran-Venugopalan model [30, 31, 54, 55, 56, 57, 58, 59], though our results would

[1] It is interesting to point out that in approaching the asymptotics of Eqs. (25) and (26) both ϵ and p_3 oscillate, such that $\epsilon/3$ becomes temporarily comparable to p_3 at proper time $\tau \sim 1/Q_s$. While the mathematical origin of these oscillations is clearly due to the Bessel functions in Eq. (24), their physical interpretation (if it exists) is presently unclear.

not depend much on this assumption[2]. In the saturation scenario, the gluon fields at the early times with $\tau \sim 1/Q_s$ are strong

$$A_\mu^a \sim \frac{Q_s}{g}. \tag{27}$$

In calculating corresponding field strength tensor $F_{\mu\nu}^a$, one would require both the Abelian and the non-Abelian parts of it. However, as classical fields and their energy density (25) decrease with proper time, for $\tau \gtrsim 1/Q_s$ the Abelian part of $F_{\mu\nu}^a$ would dominate. This is also true for quantum corrections to classical fields. Therefore, in the following discussion we will first restrict ourselves to calculating the Abelian part of $T_{\mu\nu}$ only, and will later show that inclusion of non-Abelian parts of $T_{\mu\nu}$ would not change our argument.

The most general gluon field generated through any-order Feynman diagrams in $\partial_\mu A^\mu = 0$ covariant gauge can be written as

$$A_\mu^a(x) = -i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 + i\epsilon k_0} J_\mu^a(k), \tag{28}$$

where we are using the retarded regularization for the outgoing gluon propagator $-i/k^2$ to ensure causality. The function $J_\mu^a(k)$ denotes the rest of the diagram (the truncated part). In general $J_\mu^a(k)$ is an abbreviated notation for $J_\mu^a(k; q_1, \lambda_1; q_2, \lambda_2; \ldots)$, which depends on momenta q_i of extra gluons (or quarks) in the final state and on their polarizations (helicities) λ_i. In case of the classical gluon field there are no extra particles in the final state and momenta q_i's do not enter the expression (28). (In fact, the possible particles in the final state for classical fields can be removed by using retarded regularization of gluon propagators [75].) Quantum corrections to the classical gluon field would inevitably bring in extra final state particles. The resulting "quantum" field $A_\mu^a(x)$ in Eq. (28) would also depend on momenta q_i: again we suppress this dependence in the notation. Indeed, once quantum corrections are included there is no dominant gluon field anymore: in that sense the gluon field in Eq. (28) is not really a field, but more like a scattering amplitude (located on one side of the cut), with one (k) of the many outgoing particle lines (q_i's) being off-mass shell with its propagator ending at a space-time point x_μ.

Substituting the field from Eq. (28) into the expression for the energy-momentum tensor (5)

$$\langle T^{\mu\nu} \rangle = \left\langle -F^{a\,\mu\rho}\, F^{a\,\nu}{}_{\rho} + \frac{1}{4}\, g^{\mu\nu}\, (F_{\rho\sigma}^a)^2 \right\rangle,$$

and keeping only the Abelian parts of $F_{\mu\nu}^a$'s we obtain

$$\begin{aligned} T_{\mu\nu} = \int \frac{d^4k\, d^4k'}{(2\pi)^8} \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k'_0)} \Big\langle & - [k_\mu J^{a\,\rho}(k) - k^\rho J_\mu^a(k)]\, [k'_\nu J_\rho^a(k') - k'_\rho J_\nu^a(k')] \\ & + \frac{1}{4}\, g_{\mu\nu}\, [k^\rho J^{a\,\sigma}(k) - k^\sigma J^{a\,\rho}(k)]\, [k'_\rho J_\sigma^a(k') - k'_\sigma J_\rho^a(k')] \Big\rangle, \end{aligned} \tag{29}$$

[2]There is a common misconception in the community that in McLerran-Venugopalan model one assumes that $y = \eta$: while this assumption was made in the original works on the subject [54, 30], it is actually not necessary, with all the results of McLerran-Venugopalan model easily derivable without making any assumptions on correlations between η and y (see [55]).

where the brackets $\langle \ldots \rangle$ are defined by Eq. (6) and now also include integration over all momenta q_i's. The gluon field in Eq. (28) is generated by the color sources in two colliding nuclei, which are modeled by valence quarks, just like in McLerran-Venugopalan model [30]. Of course, the resulting gluon field from Eq. (28) is not necessarily classical, it includes extra quark and gluon emissions as well as loops, just like any production diagram with incoming valence quarks of the nuclei providing the initial condition for the scattering process.

Since performing the transverse averaging over a very large nucleus in the brackets on the right hand side of Eq. (29) puts $\underline{k} = -\underline{k}'$, which results from transverse translational invariance of gluon production, we can rewrite it as

$$T_{\mu\nu} = \int \frac{d^4k\, d^4k'}{(2\pi)^8} \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k'_0)} \bigg\langle\!\!\bigg\langle - \left[k_\mu J^{a\,\rho}(k) - k^\rho J^a_\mu(k)\right] \left[k'_\nu J^a_\rho(k') - k'_\rho J^a_\nu(k')\right]$$
$$+ \frac{1}{4}\, g_{\mu\nu} \left[k^\rho J^{a\,\sigma}(k) - k^\sigma J^{a\,\rho}(k)\right] \left[k'_\rho J^a_\sigma(k') - k'_\sigma J^a_\rho(k')\right] \bigg\rangle\!\!\bigg\rangle \frac{(2\pi)^2}{S_\perp}\, \delta(\underline{k} + \underline{k}'), \tag{30}$$

where the double brackets $\langle\langle \ldots \rangle\rangle$ denote now the color averaging, integration over q_i's, summation over nucleons in the nuclei and averaging over longitudinal and remaining transverse coordinates. $S_\perp$ is the cross sectional area of the nuclei which we assume to be identical.

Let us define the following correlation function

$$D_{\mu\nu} \equiv \langle\langle J_\mu(k)\, J_\nu(k') \rangle\rangle \Big|_{\underline{k} = -\underline{k}'}. \tag{31}$$

Using the covariant gauge condition $\partial_\mu A^\mu = 0$, which translates into $k_\mu\, J^\mu(k) = k'_\mu\, J^\mu(k') = 0$, along with the $k_+ \leftrightarrow k_-$ (and $k'_+ \leftrightarrow k'_-$) symmetry of the collision, we can derive the following relations between different components of $D_{\mu\nu}$:

$$\begin{aligned}
D_{+i} &= \frac{k_j}{2\,k_-}\, D_{ji} & \qquad D_{-i} &= \frac{k_j}{2\,k_+}\, D_{ji} \\
D_{i+} &= \frac{k'_j}{2\,k'_-}\, D_{ij} & \qquad D_{i-} &= \frac{k'_j}{2\,k'_+}\, D_{ij}
\end{aligned}$$
$$\frac{D_{++}}{k_+\, k'_+} = \frac{D_{--}}{k_-\, k'_-}, \tag{32}$$

where the Latin indices $i, j = 1, 2$ indicate the transverse components of the correlators and two lowercase repeated Latin indices indicate contraction over that index.

We are interested in calculating the energy density ϵ given by (see Eq. (20))

$$\epsilon = \frac{1}{2} \left(\frac{\tau}{x_+}\right)^2 T_{++} + T_{+-}. \tag{33}$$

Using Eq. (5) we write

$$T_{++} = -\left\langle F^{a\,\rho}_{+}\, F^a_{+\rho} \right\rangle = \left\langle F^a_{+i}\, F^a_{+i} \right\rangle \tag{34}$$

and

$$T_{+-} = \frac{1}{2} \left\langle F^a_{+-}\, F^a_{+-} \right\rangle + \frac{1}{4} \left\langle F^a_{ij}\, F^a_{ij} \right\rangle. \tag{35}$$

Using Eq. (28) together with relations from Eq. (32) in Eqs. (34) and (35) we obtain

$$T_{++} \;=\; \int \frac{d^4k\, d^4k'}{(2\pi)^8}\, \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)} \left[k_+\, k'_+\, D_{ii} - \underline{k}^2\, D_{++} - \frac{1}{2}\left(\frac{k'_+}{k_-}+\frac{k_+}{k'_-}\right) k_i\, k_j\, D_{ij}\right]$$
$$\times\, \frac{(2\pi)^2}{S_\perp}\, \delta(\underline{k}+\underline{k}') \tag{36}$$

and

$$T_{+-} \;=\; \int \frac{d^4k\, d^4k'}{(2\pi)^8}\, \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)} \left[-\frac{1}{2}\,\underline{k}^2\, D_{ii} + 2k_-\, k'_-\, D_{++} + k_i\, k_j\, D_{ij}\right]$$
$$\times\, \frac{(2\pi)^2}{S_\perp}\, \delta(\underline{k}+\underline{k}'). \tag{37}$$

Substituting Eqs. (36) and (37) into Eq. (33) yields

$$\epsilon \;=\; \int \frac{d^4k\, d^4k'}{(2\pi)^8}\, \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)} \left\{\left[\frac{1}{2}\left(\frac{\tau}{x_+}\right)^2 k_+\, k'_+ - \frac{1}{2}\,\underline{k}^2\right] D_{ii} + \right.$$
$$\left. + \left[2\, k_-\, k'_- - \frac{1}{2}\left(\frac{\tau}{x_+}\right)^2 \underline{k}^2\right] D_{++} + \left[-\frac{1}{4}\left(\frac{\tau}{x_+}\right)^2\left(\frac{k'_+}{k_-}+\frac{k_+}{k'_-}\right)+1\right] k_i\, k_j\, D_{ij}\right\} \frac{(2\pi)^2}{S_\perp}\, \delta(\underline{k}+\underline{k}'). \tag{38}$$

Since the tensor structure of the correlators $D_{\mu\nu}$ from Eq. (31) is symmetric under $k \leftrightarrow k'$, and using the last relation in Eq. (32), without any loss of generality one can write

$$D_{++} \;=\; k_+\, k'_+\, f_2(k^2, k'^2, k\cdot k', k_T), \tag{39}$$

where $f_2(k^2, k'^2, k\cdot k', k_T)$ is some unknown boost-invariant function, which, due to rapidity independence of the problem, depends only on k^2, k'^2, $k\cdot k'$ and on the magnitude of the transverse momentum k_T. In general, dependence of f_2 on $k\cdot k'$ might lead to rapidity dependence: however, as we will see below the resulting leading energy density is still boost invariant. $f_2(k^2, k'^2, k\cdot k', k_T)$ is symmetric under the interchange $k^2 \leftrightarrow k'^2$. Similarly

$$D_{ii} \;=\; f_1(k^2, k'^2, k\cdot k', k_T) \tag{40}$$

and

$$k_i\, k_j\, D_{ij} \;=\; f_3(k^2, k'^2, k\cdot k', k_T) \tag{41}$$

with f_1 and f_3 also some symmetric functions under $k^2 \leftrightarrow k'^2$. Using these redefinitions we can rewrite Eq. (38) as

$$\epsilon \;=\; \int \frac{d^4k\, d^4k'}{(2\pi)^8}\, \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)} \left\{\left[\frac{1}{2}\left(\frac{\tau}{x_+}\right)^2 k_+\, k'_+ - \frac{1}{2}\,\underline{k}^2\right] f_1(k^2, k'^2, k\cdot k', k_T) + \right.$$

$$+ \left[2\, k_-\, k'_- - \frac{1}{2}\left(\frac{\tau}{x_+}\right)^2 \underline{k}^2\right] k_+\, k'_+\, f_2(k^2, k'^2, k\cdot k', k_T) +$$

$$+\left[-\frac{1}{4}\left(\frac{\tau}{x_+}\right)^2\left(\frac{k'_+}{k_-}+\frac{k_+}{k'_-}\right)\,+\,1\right]\,f_3(k^2,k'^2,k\cdot k',k_T)\Bigg\}\,\frac{(2\pi)^2}{S_\perp}\,\delta(\underline{k}+\underline{k}'). \tag{42}$$

For reasons which will become apparent in a moment, we are interested in determining the following combination of f's

$$f_1(k^2=0,k'^2=0,k\cdot k'=0,k_T)-k_T^2 f_2(k^2=0,k'^2=0,k\cdot k'=0,k_T)$$
$$-\frac{2}{k_T^2}\,f_3(k^2=0,k'^2=0,k\cdot k'=0,k_T). \tag{43}$$

To calculate it we compare Eq. (36) with

$$T_{++}\;=\;\int\frac{d^4k\,d^4k'}{(2\pi)^8}\,\frac{e^{-ik\cdot x-ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)}\,\Big\langle\Big\langle-\,[k_+J^{a\,\rho}(k)-k^\rho J^a_+(k)]$$
$$\times\;[k'_+J^a_\rho(k')-k'_\rho J^a_+(k')]\Big\rangle\Big\rangle\,\frac{(2\pi)^2}{S_\perp}\,\delta(\underline{k}+\underline{k}'), \tag{44}$$

which follows from Eq. (29). Equating the integrands of Eqs. (36) and (44) we derive

$$k_+\,k'_+\,f_1(k^2,k'^2,k\cdot k',k_T)-\underline{k}^2\,k_+\,k'_+\,f_2(k^2,k'^2,k\cdot k',k_T)-$$
$$-\frac{1}{2}\left(\frac{k'_+}{k_-}+\frac{k_+}{k'_-}\right)\,f_3(k^2,k'^2,k{\cdot}k',k_T)\;=\;\Big\langle\Big\langle-[k_+J^{a\,\rho}(k)-k^\rho J^a_+(k)]\,[k'_+J^a_\rho(k')-k'_\rho J^a_+(k')]\Big\rangle\Big\rangle. \tag{45}$$

Putting $k=-k'$ and $k^2=k'^2=0$ in Eq. (45) and employing the fact that in covariant gauge $k^\rho J^a_\rho(k)=0$ we obtain

$$f_1(k^2=0,k'^2=0,k\cdot k'=0,k_T)-k_T^2 f_2(k^2=0,k'^2=0,k\cdot k'=0,k_T)-$$
$$-\frac{2}{k_T^2}\,f_3(k^2=0,k'^2=0,k\cdot k'=0,k_T)\;=\;-\Big\langle\Big\langle J^{a\,\rho}(k)\,J^a_\rho(-k)\Big\rangle\Big\rangle\Big|_{k^2=0}. \tag{46}$$

Finally, since in order to construct the amplitude out of the field given by Eq. (28) one needs to truncate the field and put the outgoing gluon's momentum on the mass shell, $k^2=0$, we see that $J^{a\,\rho}(k)$ at $k^2=0$ is nothing but a production amplitude for a real gluon carrying momentum k (without convolution with the polarization vector). The corresponding multiplicity distribution of the produced gluons is given by

$$\frac{dN}{d^2k\,dy}\;=\;\frac{1}{2(2\pi)^3}\,\Big\langle\Big\langle J^{a\,\rho}(k)\,J^a_\rho(-k)\Big\rangle\Big\rangle\Big|_{k^2=0}. \tag{47}$$

Therefore,

$$f_1(k^2=0,k'^2=0,k\cdot k'=0,k_T)-k_T^2 f_2(k^2=0,k'^2=0,k\cdot k'=0,k_T)$$
$$-\frac{2}{k_T^2}\,f_3(k^2=0,k'^2=0,k\cdot k'=0,k_T)\;=\;-2(2\pi)^3\,\frac{dN}{d^2k\,dy} \tag{48}$$

and, for a cylindrical nucleus,

$$\frac{1}{S_\perp}\Big[f_1(k^2=0,k'^2=0,k\cdot k'=0,k_T)-k_T^2 f_2(k^2=0,k'^2=0,k\cdot k'=0,k_T)$$

$$-\frac{2}{k_T^2}\,f_3(k^2=0,k'^2=0,k\cdot k'=0,k_T)\Big] \;=\; -2(2\pi)^3\,\frac{dN}{d^2k\,dy\,d^2b}. \tag{49}$$

Now let us get back to Eq. (42). Rewriting for each of the f's

$$f_i(k^2,k'^2,k\cdot k',k_T) \;=\; f_i(k^2=0,k'^2=0,k\cdot k'=0,k_T)+$$

$$+\left[f_i(k^2,k'^2,k\cdot k',k_T)-f_i(k^2=0,k'^2=0,k\cdot k'=0,k_T)\right] \tag{50}$$

and keeping only the $f_i(k^2=0,k'^2=0,k\cdot k'=0,k_T)$ in Eq. (42) we can perform the longitudinal momenta (k_+,k_-,k'_+,k'_-) integrations with the help of Eq. (B7) from Appendix B obtaining

$$\epsilon \;\approx\; -\frac{1}{8\,S_\perp}\int\frac{d^2k}{(2\,\pi)^2}\,k_T^2\,\left\{[J_1(k_T\tau)]^2+[J_0(k_T\tau)]^2\right\}$$

$$\times\Big[f_1(k^2=0,k'^2=0,k\cdot k'=0,k_T)-k_T^2 f_2(k^2=0,k'^2=0,k\cdot k'=0,k_T)$$

$$-\frac{2}{k_T^2}\,f_3(k^2=0,k'^2=0,k\cdot k'=0,k_T)\Big], \tag{51}$$

which, after employing Eq. (49) becomes

$$\epsilon \;\approx\; \frac{\pi}{2}\int d^2k\,\frac{dN}{d^2k\,d\eta\,d^2b}\;k_T^2\,\left\{[J_1(k_T\tau)]^2+[J_0(k_T\tau)]^2\right\}. \tag{52}$$

(Again we have used the rapidity-independence of the gluon spectrum $\frac{dN}{d^2k\,d\eta\,d^2b}$ to replace y with η.)

One might worry that the functions $f_i(k^2,k'^2,k\cdot k',k_T)$ may not have a finite $k^2,k'^2,k\cdot k'\to 0$ limit, which would be dangerous for the decomposition of Eq. (50) [72]. However, let us remind the reader that the quantity $J_\mu^a(k)$ defined in Eq. (28) has the meaning of (truncated) gluon production amplitude for the off-shell gluon with virtuality k^2. In the $k^2\to 0$ limit $J_\mu^a(k)$ becomes the gluon production amplitude for an on-shell gluon, and is indeed finite. Therefore, the correlation functions $D_{\mu\nu}$ from Eq. (31), which in the $k^2,k'^2,k\cdot k'\to 0$ limit have the meaning of the gluon production amplitude squared (but without the Lorentz index contraction), are also finite in this limit. This implies that the functions $f_i(k^2,k'^2,k\cdot k',k_T)$, defined in terms of various components of $D_{\mu\nu}$ in Eqs. (39), (40) and (41), are finite in the $k^2,k'^2,k\cdot k'\to 0$ limit.

For the proper time τ much larger than $1/\langle k_T\rangle$, with k_T the typical transverse momentum in the distribution $\frac{dN}{d^2k\,d\eta\,d^2b}$, Eq. (52) becomes

$$\epsilon\Big|_{\tau\gg 1/\langle k_T\rangle} \;\approx\; \frac{1}{\tau}\int d^2k\,\frac{dN}{d^2k\,d\eta\,d^2b}\;k_T \;=\; \frac{1}{\tau}\,\frac{dE_T}{d\eta\,d^2b}, \tag{53}$$

i.e., it falls off as $1/\tau$.

Therefore, we have shown that the energy density ϵ of a gluon field produced in a heavy ion collision always has a non-zero term scaling as $\sim 1/\tau$. However, to demonstrate that this term dominates at late times, we still need to prove that it does not get canceled by the terms we left out in writing down the decomposition of Eq. (50) and keeping the first terms only. Thus we have to analyze the contribution arising from substituting the terms from the square brackets of Eq. (50) into Eq. (42). We need to show that such contributions fall off faster than $1/\tau$, and, therefore, can be neglected at late times. Here we will demonstrate that this is true for one of the terms — the f_1-term in Eq. (42). The proof for the other two terms on the right hand side of Eq. (42) would be analogous.

Substituting the square brackets from Eq. (50) into Eq. (42) we obtain the following contribution to the energy density, which we have to prove to be small:

$$\frac{1}{2} \int \frac{d^4k\, dk'_+\, dk'_-}{(2\pi)^6\, S_\perp} \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k'_0)} \left[\left(\frac{\tau}{x_+}\right)^2 k_+\, k'_+ - \underline{k}^2 \right]$$
$$\times\, [f_1(k^2, k'^2, k\cdot k', k_T) - f_1(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T)]. \tag{54}$$

For a wide range of amplitudes one can write

$$f_1(k^2, k'^2, k\cdot k', k_T) - f_1(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T) \;=\; (k^2\, k'^2)^{\Delta_1}\, g^{(1)}(k^2, k'^2, k\cdot k', k_T) +$$
$$+\, [(k + k')^2]^{\Delta_2}\, g^{(2)}(k^2, k'^2, k\cdot k', k_T), \tag{55}$$

where $g^{(1)}(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T) \neq 0$, $g^{(2)}(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T) \neq 0$, and $\Delta_1, \Delta_2 > 0$. In arriving at Eq. (55) we have also used the fact that $f_1(k^2, k'^2, k\cdot k', k_T) = f_1(k'^2, k^2, k\cdot k', k_T)$, which follows from the $k \leftrightarrow k'$ symmetry in the definition of $f_1(k^2, k'^2, k\cdot k', k_T)$ given by Eq. (40) along with Eq. (31). In Eq. (55) we put a power of $(k + k')^2$ instead of a power of $k\cdot k'$ in front of $g^{(2)}$. Similarly to Eq. (50) we write

$$g^{(i)}(k^2, k'^2, k\cdot k', k_T) \;=\; g^{(i)}(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T) +$$
$$+\, [g^{(i)}(k^2, k'^2, k\cdot k', k_T) - g^{(i)}(k^2 = 0, k'^2 = 0, k\cdot k' = 0, k_T)] \tag{56}$$

for $i = 1, 2$. Substituting the first term on the right hand side of Eq. (56) for $g^{(1)}$ into the first term on the right hand side of Eq. (55), and then plugging the resulting contribution into Eq. (54) yields

$$\frac{1}{2} \int \frac{d^4k\, dk'_+\, dk'_-}{(2\pi)^6\, S_\perp} \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k'_0)} \left\{ \left(\frac{\tau}{x_+}\right)^2 k_+\, k'_+ - \underline{k}^2 \right\} (k^2\, k'^2)^{\Delta_1}\, g^{(1)}(0, 0, 0, k_T). \tag{57}$$

Performing the k_+, k_-, k'_+, k'_- integrations in Eq. (57) with the help of Eq. (A1) in Appendix A and Eq. (B5) with $\lambda = 1$ in Appendix B we obtain

$$\frac{e^{2\pi i \Delta_1}}{8\, S_\perp\, \Gamma(1 - \Delta_1)^2} \int \frac{d^2k}{(2\pi)^2}\, g^{(1)}(0, 0, 0, k_T)\, k_T^2 \left(\frac{2\, k_T}{\tau}\right)^{2\,\Delta_1} \left\{ [J_{-\Delta_1 - 1}(k_T \tau)]^2 + [J_{-\Delta_1}(k_T \tau)]^2 \right\}, \tag{58}$$

which, for $\tau \gg 1/\langle k_T \rangle$, scales as

$$\sim \frac{1}{\tau^{1 + 2\Delta_1}}, \tag{59}$$

and is, therefore, negligibly small at late proper times compared to the leading contribution to energy density given by Eq. (53). (Here we assume that the typical transverse momentum $\langle k_T \rangle$ is the same for $\frac{dN}{d^2k\, d\eta\, d^2b}$ in Eq. (52) and for $g^{(1)}(0,0,k_T)$ in Eq. (58): both functions result from expanding the same amplitude in powers of $k^2\, k'^2$, and no new scale can arise from such an expansion, which justifies our assumption.) The particular way of regularizing the k^2 branch cut used in Eq. (A1) and Eq. (B1) is not essential for arriving at Eq. (59), since other regularizations would yield the same result.

The second term on the right hand side of Eq. (55) gives a similarly small contribution. To see this we substitute the first term on the right hand side of Eq. (56) for $g^{(2)}$ into the second term on the right hand side of Eq. (55), and then substitute the result into Eq. (54) obtaining

$$\int \frac{d^4k\, dk'_+\, dk'_-}{(2\pi)^6\, 2\, S_\perp} \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k'_0)} \left\{ \left(\frac{\tau}{x_+}\right)^2 k_+\, k'_+ - \underline{k}^2 \right\} [(k+k')^2]^{\Delta_2}\, g^{(2)}(0,0,0,k_T) =$$

$$= [-\partial_\mu \partial^\mu]^{\Delta_2} \frac{1}{8\, S_\perp} \int \frac{d^2k}{(2\pi)^2}\, g^{(2)}(0,0,0,k_T)\, k_T^2\, \left\{ [J_1(k_T\tau)]^2 + [J_0(k_T\tau)]^2 \right\}. \tag{60}$$

For $\tau \gg 1/\langle k_T \rangle$ the integral on the right of Eq. (60) scales as $\sim 1/\tau$: applying the derivatives we see that the whole expression in Eq. (60) scales as

$$\sim \frac{1}{\tau^{1+2\Delta_2}}, \tag{61}$$

and is also negligibly small at late proper times compared to the leading contribution to energy density given by Eq. (53).

For the second term on the right hand side of Eq. (56), $g^{(i)}(k^2, k'^2, k\cdot k', k_T) - g^{(i)}(0,0,0,k_T)$ with $i = 1$(or $i = 2$), one can repeat the procedure outlined above for $f_1(k^2, k'^2, k\cdot k', k_T) - f_1(0,0,0,k_T)$, using the redefinition just like in Eq. (55) and showing that the leading term in the resulting decomposition, similar to Eq. (56), falls off faster with τ than Eq. (59) (or Eq. (61)). Iterating the procedure would generate a series of corrections falling off at progressively higher powers of $1/\tau$, all of which could be neglected at $\tau \gg 1/\langle k_T \rangle$.

Of course the assumption of Eq. (55), while quite general, does not include all the possibilities. One might imagine other ways for $f_1(k^2, k'^2, k\cdot k', k_T) - f_1(0,0,0,k_T)$ to approach zero as $k^2, k'^2, k\cdot k' \to 0$: it might scale as $1/(\ln k^2\, \ln k'^2)$, or, less likely, as $e^{-k_T^2/k^2 - k_T^2/k'^2}$. In any case, Eq. (A1) suggests that each power of k^2 (or each power of k'^2 or of $k\cdot k'$) gives a power of $1/\tau$ for energy density ϵ in coordinate space: $k^2 \to 1/\tau$. (Indeed the powers of k_T do not modify the τ-dependence of $T_{\mu\nu}$ at all.) Therefore, one may argue that after the momentum integration is done in Eq. (52), the $f_1(k^2, k'^2, k\cdot k', k_T) - f_1(0,0,0,k_T)$ term, when substituted into Eq. (42), yields approximately the following contribution to ϵ

$$\frac{1}{\tau} \left[f_1\left(\frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}, \langle k_T \rangle\right) - f_1(0,0,0,\langle k_T \rangle) \right], \tag{62}$$

which falls off faster than $1/\tau$ and can thus be neglected compared to Eq. (53). This conclusion is natural, since the term in Eq. (62), or, equivalently, the second term on the right hand side of Eq. (50), does not contribute to the production cross section, as follows from Eq. (48), which is another way of saying that it is not important at late times.

The proofs that the contributions to energy density ϵ generated by substituting $f_2(k^2, k'^2, k \cdot k', k_T) - f_2(0,0,0,k_T)$ and $f_3(k^2, k'^2, k \cdot k', k_T) - f_3(0,0,0,k_T)$ instead of f_2 and f_3 into Eq. (42) are also subleading at large τ can be constructed by analogy to the above.

Finally, we have to comment on our use of the Abelian part of $T_{\mu\nu}$ only in Eq. (29) and throughout this Section. Including the non-Abelian parts of the field strength tensor $F_{\mu\nu}$ would generate higher powers of A_μ^a in the definition (5) of $T_{\mu\nu}$. Using Eq. (28) those extra powers can be rewritten as extra integrals over k'' and k''' in the extra terms which would be added to Eq. (29). Due to Eq. (A1), each of these extra integrals would (at least) generate a Bessel function $J_{-\Delta}(k_T\,\tau)$, which at large τ scales as $(1/\sqrt{\tau})\,\cos(k_T\,\tau + \frac{\pi}{2}\Delta - \frac{\pi}{4})$. Even without the cosine, one can immediately see that the cubic in A_μ^a term in $T_{\mu\nu}$ would fall off at least like $1/\tau^{3/2}$ at large τ. The quadric terms would fall off at least like $1/\tau^2$. Both of these terms would be negligibly small compared to the leading quadratic term scaling as $1/\tau$ shown in Eq. (53).

Eq. (53) has a straightforward physical interpretation. Every Feynman diagram has a final state in which the particles are propagating as non-interacting plane waves until the infinite late times. Indeed the energy density of such a "free-streaming" state scales as $\sim 1/\tau$, and this is exactly what Eq. (53) represents.

Therefore, in this Section we have proven that in the rapidity-independent case, defined by Eqs. (12) and (13) for the total rapidity interval in the collision of two very large nuclei, at any order in the perturbative expansion in the strong coupling g, the resulting gluon field's energy density has a non-vanishing term which is dominant at late times giving $\epsilon \sim 1/\tau$ (53). Hence it appears that, in this boost-invariant case, thermalization leading to Bjorken hydrodynamic description of the evolution of produced gluonic system, can not result from resummation of perturbative QCD diagrams.

4 Generalization to the Rapidity-Dependent Case

For rapidity intervals $Y \gtrsim \frac{1}{\alpha_s}$ in heavy ion collisions the quantum evolution corrections [73, 33, 34, 35] would become important making the produced particle distribution rapidity dependent. Below we are first going to argue that rapidity-dependent hydrodynamic description may only change the $\epsilon \sim 1/\tau^{4/3}$ scaling of the ideal Bjorken energy density to a higher power, $\epsilon \sim 1/\tau^{4/3+\Delta}$. We will then demonstrate that the rapidity-dependent quantum corrections, such as the ones introduced by the BFKL evolution [73], would *not* modify the $\epsilon \sim 1/\tau$ scaling derived in the previous Section.

4.1 Rapidity-Dependent Hydrodynamics

In the rapidity dependent case the most general form of the energy-momentum tensor is given by the equation similar to Eq. (16)

$$T_{\mu\nu} \;=\; A(\tau,\eta)\,u_\mu u_\nu + B(\tau,\eta)\,(u_\mu v_\nu + u_\nu v_\mu) + C(\tau,\eta)\,v_\mu v_\nu + D(\tau,\eta)\,g_{\mu\nu}, \tag{63}$$

with u_μ and v_μ still given by Eqs. (14) and (15) and where now all the coefficients A, B, C, D are also functions of the space-time rapidity η. Due to this η-dependence the $+ \leftrightarrow -$ symmetry argument no longer applies in general. However, it still holds at mid-rapidity ($\eta = 0$) for a

collision of two identical nuclei leading to

$$B(\tau, \eta = 0) \, = \, 0. \tag{64}$$

Applying the conservation of energy-momentum tensor condition (22) to the tensor in Eq. (63) yields

$$\begin{aligned} \tau\, \partial_\tau B - 2\, \partial_\eta D + 2\, \partial_\eta C + 2\, B \;&=\; 0 \\ 2\, \tau\, \partial_\tau A + \partial_\eta B + 2\, \tau\, \partial_\tau D + 2\, A + 2\, C \;&=\; 0. \end{aligned} \tag{65}$$

The energy-momentum tensor in Eq. (63) would describe a hydrodynamic system if it could be reduced to the standard hydrodynamic form

$$T_{\mu\nu} \, = \, (\epsilon + p)\, w_\mu\, w_\nu - p\, g_{\mu\nu}, \tag{66}$$

where w_μ is the four-vector of the fluid velocity, $w_\mu\, w^\mu = 1$. For the 1+1-dimensional expansion of the system created in a collisions of two very large nuclei considered here the fluid velocity has zero transverse component, $\underline{w} = 0$, such that $w_\mu = (w_+, w_-, \underline{0})$. Matching Eq. (63) onto Eq. (66) we obtain

$$A \, = \, \epsilon + p + C \tag{67}$$

and

$$D \, = \, -p. \tag{68}$$

For the hydrodynamic energy momentum tensor (66) the following relation holds

$$T_{++}\, T_{--} \, = \, (T_{+-} + p)^2, \tag{69}$$

leading to a constraint

$$C \, = \, \frac{B^2}{4\, A}. \tag{70}$$

Combining Eq. (65) and Eq. (70) with the equation of state relating ϵ and p, would give us a complete set of rapidity-dependent hydrodynamic equations. However the resulting system of equations is nonlinear and is hard to solve analytically. Instead we are going to construct a perturbative solution for small rapidity-dependent corrections to Bjorken hydrodynamics [1]. We begin by noting that, since $B = 0$ in the boost-invariant case considered in the previous Section, we can assume that non-zero B reflects the deviation from the ideal Bjorken hydrodynamics, and could be assumed small if we are interested in small corrections to the latter. Assuming that $B \ll A$ and keeping only linear in B corrections allows us to neglect C, since, due to Eq. (70), $C \sim B^2$. Than, using Eqs. (67) and (68) in Eq. (65) yields

$$\begin{aligned} \tau\, \partial_\tau B + 2\, \partial_\eta p + 2\, B \;&=\; 0 \\ 2\, (\tau\, \partial_\tau \epsilon + \epsilon + p) + \partial_\eta B \;&=\; 0. \end{aligned} \tag{71}$$

We are interested in the solution for the ideal gas equation of state: therefore we put $\epsilon = 3\, p$. The most general solution of Eq. (71) satisfying the condition of Eq. (64) and mapping back onto Bjorken hydrodynamic behavior for small B is

$$\epsilon \, = \, \epsilon_0\, \cos(\sqrt{\Delta}\, \eta)\, \frac{1}{\tau^{\frac{1}{3}\,(5 - \sqrt{1 - 3\Delta})}} \tag{72}$$

with

$$B = -\frac{2}{3}\,\epsilon_0\,\frac{\sqrt{1-3\Delta}-1}{\sqrt{\Delta}}\,\sin(\sqrt{\Delta}\,\eta)\,\frac{1}{\tau^{\frac{1}{3}\,(5-\sqrt{1-3\Delta})}}, \tag{73}$$

where Δ and ϵ_0 are arbitrary constants. The corresponding flow velocity components are given by

$$w_\pm \approx \frac{x_\pm}{\tau}\left(1 \pm \frac{3\,B}{8\,\epsilon}\right). \tag{74}$$

Looking at the solution given by Eq. (72) one may wonder why the energy density is not positive definite. Indeed for $\Delta < 0$ the energy density ϵ from Eq. (72) becomes positive definite, since $\cos(\sqrt{\Delta}\,\eta)$ would be replaced by $\cosh(\sqrt{|\Delta|}\,\eta)$. However, the resulting rapidity distribution of energy density would increase as one moves further away from mid-rapidity, which is unphysical. Therefore one has to have $\Delta > 0$. Resolution of the positivity problem for ϵ comes from the necessity to satisfy the $B \ll A$ assumption which we have made at the beginning of this calculation. It translates into $B \ll \epsilon$ condition, which is satisfied by Eqs. (72) and (73) only if $\sqrt{\Delta}\,\eta \ll 1$. Since in this Section we are interested in large rapidity intervals, $\eta \sim Y \gtrsim 1/\alpha_s$, the $\sqrt{\Delta}\,\eta \ll 1$ requires that $\Delta \lesssim \alpha_s \ll 1$. Hence, for large rapidities, Eqs. (72) and (73) are valid only at the lowest order in Δ

$$\epsilon \approx \frac{\epsilon_0}{\tau^{\frac{4}{3}+\frac{\Delta}{2}}}\left(1 - \frac{1}{2}\,\Delta\,\eta^2\right), \tag{75}$$

$$B \approx \frac{\epsilon_0}{\tau^{\frac{4}{3}+\frac{\Delta}{2}}}\,\Delta\,\eta, \tag{76}$$

where we did not expand $\tau^{-\Delta/2}$ since, at late times, $\Delta\,\ln\tau$ does not have to be small for $B \ll \epsilon$ condition to hold. For small $\sqrt{\Delta}\,\eta$ the energy density in Eq. (75) is indeed positive.

Eq. (75) has an important feature which we would like to emphasize: since $\Delta > 0$, it shows that the energy density of the boost-non-invariant ideal hydrodynamics falls off with τ *faster* than the energy density of the boost-invariant ideal Bjorken hydrodynamics [1]. Here we have proven it only for a small rapidity-dependent perturbation of the Bjorken solution. However one should expect our conclusion to hold in a general case of a rapidity-dependent hydrodynamics. In the case of a rapidity-dependent hydrodynamics, the longitudinal pressure is higher than in the boost-invariant Bjorken case, generating the longitudinal acceleration of the flow (see e.g. Eq. (74)). The central-rapidity high-density system starts expanding faster than in Bjorken case, leading to a faster depletion of the energy density with τ. In other words, once the longitudinal homogeneity of pure Bjorken hydrodynamics is broken by some rapidity-dependent phenomena, the system starts doing more work in the longitudinal direction than it was doing in pure Bjorken hydrodynamics case, and this leads to a faster decrease of energy density with proper time.[3]

4.2 Rapidity-Dependent Energy Density

Here we are going to generalize the argument of Sect. 3.3 to the case of rapidity-dependent gluon fields. It is impossible to define co-moving energy density for a general energy-momentum tensor like the one given in Eq. (63), since, in the general not necessarily hydrodynamic case,

[3] The author would like to thank Ulrich Heinz for explaining to him this argument.

one can not define the co-moving frame, and in the case of hydrodynamics (66) one needs to know the flow velocity to define the co-moving frame, which is impossible to do without solving the hydrodynamics equations (65). Therefore we will restrict our analysis to the case of mid-rapidity, $\eta = 0$, where, for a collision of two identical nuclei, the co-moving frame is just the center of mass frame of the two nuclei. There Eq. (33) would apply, such that

$$\epsilon(\tau, \eta = 0) \, = \, \frac{1}{2} \left(\frac{\tau}{x_+} \right)^2 T_{++}(\tau, \eta = 0) + T_{+-}(\tau, \eta = 0) \, = \, T_{++}(\tau, \eta = 0) + T_{+-}(\tau, \eta = 0). \quad (77)$$

Repeating the steps from Section 3.3 we write

$$\epsilon(\tau, \eta = 0) \, = \, \int \frac{d^4k \, d^4k'}{(2\pi)^8} \, \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k_0')}\bigg|_{\eta=0} \left\{ \left[\frac{1}{2} \left(\frac{\tau}{x_+} \right)^2 k_+ \, k'_+ - \frac{1}{2} \, \underline{k}^2 \right] \right.$$

$$\times \, f_1(k^2, k_+, k'^2, k'_+, k\cdot k', k_T) \, + \left[2\, k_- \, k'_- - \frac{1}{2} \left(\frac{\tau}{x_+} \right)^2 \underline{k}^2 \right] k_+ \, k'_+ \, f_2(k^2, k_+, k'^2, k'_+, k\cdot k', k_T)$$

$$\left. + \left[-\frac{1}{4} \left(\frac{\tau}{x_+} \right)^2 \left(\frac{k'_+}{k_-} + \frac{k_+}{k'_-} \right) + 1 \right] f_3(k^2, k_+, k'^2, k'_+, k\cdot k', k_T) \right\} \frac{(2\pi)^2}{S_\perp} \, \delta(\underline{k} + \underline{k}'). \quad (78)$$

where now, in the rapidity dependent case, f_i's are functions of $k_\pm$ and $k'_\pm$ as well. However, since we can always rewrite $k_- = (k^2 + k_T^2)/2k_+$ and $k'_- = (k'^2 + k_T^2)/2k'_+$, we put only k_+ and k'_+ in the arguments of the functions f_i.

Rapidity-dependent quantum evolution corrections come in as logarithms of Bjorken x variable [73]. If p_+ is a large longitudinal momentum carried by a nucleon in the nucleus moving in the $+$-direction, than $x = k_+/p_+$. The rapidity-dependent corrections would then bring in powers of $\alpha_s \ln 1/x = \alpha_s \ln p_+/k_+$. Resummation of such corrections for the gluon production amplitudes generates powers of $1/x$, or, equivalently, p_+/k_+. Therefore, to verify whether such corrections modify the τ-dependence of ϵ, we can consider the following general form for the functions f_i's

$$f_i(k^2, k_+, k'^2, k'_+, k\cdot k', k_T) \, = \, \left(\frac{p_+}{k_+} \frac{p_+}{k'_+} \right)^\lambda \tilde{f}_i(k^2, k'^2, k\cdot k', k_T), \quad (79)$$

where we again used the fact that f's are symmetric under $k \leftrightarrow k'$ interchange. For simplicity we assume the power λ to be the same for f_1, f_2 and f_3: this assumption is not crucial and can be easily relaxed. The logarithmic corrections to f_i's, i.e., terms with $\ln p_+/k_+$ and $\ln p_+/k'_+$, can be obtained from f_i's in Eq. (79) by differentiating it with respect to λ. Indeed that would give only logarithms like $\ln(p_+^2/k_+k'_+)$, but not $\ln k_+/k'_+$: while we are quite confident that the latter terms never appear in perturbation theory, our approach can be easily generalized to include both types of logarithms by putting different powers for p_+/k_+ and p_+/k'_+ factors in Eq. (79). (A careful reader may worry that the p_+-dependence was never explicitly shown in the argument of f_i's and was explicitly assumed there: in fact, due to boost-invariance, the p_+-dependence enters in f_i's only through the ratios of p_+/k_+ and p_+/k'_+. In the rapidity-independent case of Section 3.3, f_i's were independent of p_+, which corresponds to the eikonal limit.)

Substituting f_i's from Eq. (79) into Eq. (78) we obtain

$$\epsilon(\tau, \eta = 0) \;=\; \int \frac{d^4k\, d^4k'}{(2\pi)^8} \, \frac{e^{-ik\cdot x - ik'\cdot x}}{(k^2 + i\epsilon k_0)\,(k'^2 + i\epsilon k'_0)}\bigg|_{\eta=0} \left\{ \left[\frac{1}{2} \left(\frac{\tau}{x_+} \right)^2 k_+ \, k'_+ - \frac{1}{2} \, \underline{k}^2 \right] \right.$$

$$\times \, \tilde{f}_1(k^2, k'^2, k\cdot k', k_T) \, + \left[2\, k_- \, k'_- - \frac{1}{2} \left(\frac{\tau}{x_+} \right)^2 \underline{k}^2 \right] k_+ \, k'_+ \;\; \tilde{f}_2(k^2, k'^2, k\cdot k', k_T) +$$

$$\left. + \left[-\frac{1}{4} \left(\frac{\tau}{x_+} \right)^2 \left(\frac{k'_+}{k_-} + \frac{k_+}{k'_-} \right) + 1 \right] \tilde{f}_3(k^2, k'^2, k\cdot k', k_T) \right\} \left(\frac{p_+}{k_+} \frac{p_+}{k'_+} \right)^\lambda \frac{(2\pi)^2}{S_\perp} \, \delta(\underline{k} + \underline{k}'). \qquad (80)$$

To perform the longitudinal momentum integrals we will use the integral in Eqs. (B1) and (B6) of Appendix B. There we can see that, similar to the rapidity-independent case, each positive extra power of k^2 (or k'^2 or $k \cdot k'$) gives a power of $1/\tau$. Therefore, we again are interested in contribution of $f_i(k^2 = 0, k_+, k'^2 = 0, k'_+, k \cdot k' = 0, k_T)$ as in the terms giving the leading-τ behavior. Similar to Eq. (49) we can write

$$\frac{1}{S_\perp} \left(\frac{p_+}{k_+} \frac{p_+}{-k_+} \right)^\lambda \left[\tilde{f}_1(0,0,0,k_T) - k_T^2 \tilde{f}_2(0,0,0,k_T) - \frac{2}{k_T^2} \, \tilde{f}_3(0,0,0,k_T) \right] \;=\; -2(2\pi)^3 \, \frac{dN}{d^2k \, dy \, d^2b}. \qquad (81)$$

which shows that this combination of f_i's is not zero. Using Eq. (81) in Eq. (80), where we put $k^2 = k'^2 = 0$ in the arguments of all f_i's, and performing the longitudinal integrations using the formulas from Appendix B yields for the leading term in energy density

$$\epsilon(\tau, \eta = 0) \, \approx \, \frac{\pi}{2} \int d^2k \, \frac{dN}{d^2k \, d\eta \, d^2b}\bigg|_{\eta=0} \, k_T^2 \, \left\{ [J_{-1-\lambda}(k_T\tau)]^2 + [J_{-\lambda}(k_T\tau)]^2 \right\}. \qquad (82)$$

In arriving at Eq. (82) we have noticed that, according to Eq. (B6), each power of k_+ or k'_+ gives a power of $k_T \, e^\eta/\sqrt{2}$ after the integration. For on-mass shell gluons in Eq. (81) one has $k_+ = k_T \, e^y/\sqrt{2}$. Therefore, the powers of $k_T \, e^\eta/\sqrt{2}$ were absorbed in $\frac{dN}{d^2k \, d\eta \, d^2b}$ by just replacing $y \to \eta$.

The large-τ asymptotics of Eq. (82) is the same as in Eq. (53):

$$\epsilon(\tau, \eta = 0)\bigg|_{\tau \gg 1/\langle k_T \rangle} \, \approx \, \frac{1}{\tau} \int d^2k \, \frac{dN}{d^2k \, d\eta \, d^2b}\bigg|_{\eta=0} \, k_T \, = \, \frac{1}{\tau} \, \frac{dE_T}{d\eta \, d^2b}\bigg|_{\eta=0}. \qquad (83)$$

Therefore, we have proven that even in the rapidity-dependent case the mid-rapidity energy density given by the Feynman diagrams falls off as $1/\tau$ at large τ. This conclusion could be easily derived by just analyzing Eqs. (B1) and (B6): one can see there that each extra power of k_+ does not bring in any new powers of τ, and only modifies the order of the Bessel function, which can not change the τ-dependence, as follows from Eq. (82).

Eq. (83) shows that the scaling of energy density of the rapidity-dependent solution of the hydrodynamics equations given by Eq. (75), $\epsilon \sim 1/\tau^{\frac{4}{3}+\frac{\Delta}{2}}$, can not come from the leading contribution of Feynman diagrams. Indeed, the subleading contributions may still lead to energy density falling off with τ faster than $1/\tau$: however, due to Eq. (A1), such contributions must come in with extra positive powers of k^2 in momentum space. They would go to zero in the on-mass shell $k^2 \to 0$ limit and, thus, would not contribute to the production cross section. Such corrections are probably irrelevant for all physical observables.

5 Including Quarks

To generalize our conclusion to massless quarks we will restrict our discussion to rapidity-independent case only: generalization to the rapidity-dependent case can be easily done following the procedure outlined in Section 4.2. We start with the energy-momentum tensor for a single massless quark flavor:

$$T_{\mu\nu}^{quark} \;=\; \frac{i}{2}\,\overline{\psi}\,(\gamma_\mu\,D_\nu + \gamma_\nu\,D_\mu)\,\psi. \tag{84}$$

The corresponding energy density is given by Eq. (33), which we again want to rewrite as a double integral, just like Eq. (42), by Fourier transforming the quark field

$$\psi(x) \;=\; i\int \frac{d^4k}{(2\pi)^4}\,\frac{k\cdot\gamma}{k^2+i\epsilon k_0}\,e^{-ik\cdot x}\,\xi(k) \tag{85}$$

and

$$\overline{\psi}(x) \;=\; i\int \frac{d^4k'}{(2\pi)^4}\,\tilde{\xi}(k')\,\frac{k'\cdot\gamma}{k'^2+i\epsilon k'_0}\,e^{-ik'\cdot x} \tag{86}$$

with $\xi(k)$ and $\tilde{\xi}(k')$ some spinors. In the following, similar to Section 3.3, we will keep only the Abelian part of $T_{\mu\nu}^{quark}$. The non-Abelian corrections are suppressed at late times and can be neglected, since, just like in Section 3.3, they fall off faster than the Abelian term by at least a factor of $1/\sqrt{\tau}$. Replacing the covariant derivatives D_μ in Eq. (84) by a regular derivative ∂_μ and substituting Eqs. (85) and (86) in it we obtain

$$T_{\mu\nu}^{quark} \;=\; -\frac{1}{2}\int\frac{d^4k\,d^4k'}{(2\pi)^8}\,\frac{e^{-ik\cdot x-ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)}\,\left\langle\tilde{\xi}(k')\,k'\cdot\gamma\,(\gamma_\mu\,k_\nu+\gamma_\nu\,k_\mu)\,k\cdot\gamma\,\xi(k)\right\rangle. \tag{87}$$

Rewriting

$$\left\langle\!\left\langle\tilde{\xi}(k')\,k'\cdot\gamma\,\gamma_\mu\,k\cdot\gamma\,\xi(k)\right\rangle\!\right\rangle = -\left[(k_\mu-k'_\mu)\,h_1(k^2,k'^2,k\cdot k',k_T)+(k_\mu+k'_\mu)\,h_2(k^2,k'^2,k\cdot k',k_T)\right] \tag{88}$$

in Eq. (87) and using Eq. (33) we write

$$\begin{aligned}\epsilon^{quark}(\tau) \;=\; &\int\frac{d^4k\,d^4k'}{(2\pi)^8}\,\frac{e^{-ik\cdot x-ik'\cdot x}}{(k^2+i\epsilon k_0)\,(k'^2+i\epsilon k'_0)}\\ &\times\left\{\left[\frac{1}{2}\left(\frac{\tau}{x_+}\right)^2(k_+-k'_+)\,k_++k_+\,k_--\frac{1}{2}\,(k'_+\,k_-+k_+\,k'_-)\right]h_1(k^2,k'^2,k\cdot k',k_T)+\right.\\ &\left.+\left[\frac{1}{2}\left(\frac{\tau}{x_+}\right)^2(k_++k'_+)\,k_++k_+\,k_-+\frac{1}{2}\,(k'_+\,k_-+k_+\,k'_-)\right]h_2(k^2,k'^2,k\cdot k',k_T)\right\}\frac{(2\pi)^2}{S_\perp}\,\delta(\underline{k}+\underline{k}').\end{aligned} \tag{89}$$

Similar to Section 3.3, by putting $k=-k'$ and $k^2=k'^2=0$ in Eq. (88) we derive

$$\frac{1}{S_\perp}\,h_1(k^2=0,k'^2=0,k\cdot k'=0,k_T) \;=\; \frac{1}{S_\perp}\left\langle\!\left\langle\tilde{\xi}(-k)\,k\cdot\gamma\,\xi(k)\right\rangle\!\right\rangle\Big|_{k^2=0} \;=\; 2(2\pi)^3\,\frac{dN^q}{d^2k\,dy\,d^2b}, \tag{90}$$

where $\frac{dN^q}{d^2k\, dy\, d^2b}$ is the multiplicity of the produced quarks. Arguing, just like we did for gluons, that the leading-τ behavior for the quark energy density is given by $h_1(k^2 = 0, k'^2 = 0, k \cdot k' = 0, k_T)$ and $h_2(k^2 = 0, k'^2 = 0, k \cdot k' = 0, k_T)$ in Eq. (89) we can integrate over the longitudinal momenta in Eq. (89) using Eq. (B7) in Appendix B obtaining

$$\epsilon^{quark}(\tau) \approx \frac{\pi}{2} \int d^2k \, \frac{dN^q}{d^2k\, d\eta\, d^2b}\, k_T^2 \left\{ -J_0(k_T\tau)\, J_2(k_T\tau) + 2[J_1(k_T\tau)]^2 + [J_0(k_T\tau)]^2 \right\} +$$
$$+ \frac{1}{8\, S_\perp} \int \frac{d^2k}{(2\,\pi)^2}\, h_2(0,0,0,k_T)\, k_T^2 \left\{ -J_0(k_T\tau)\, J_2(k_T\tau) - 2[J_1(k_T\tau)]^2 + [J_0(k_T\tau)]^2 \right\}, \tag{91}$$

where we also substituted space-time rapidity η instead of y in the quark multiplicity distribution, which makes no difference in the boost invariant case we consider. We can assume that $h_2(k^2 = 0, k'^2 = 0, k \cdot k' = 0, k_T)$ is a steeply falling function of k_T for $k_T \gg \langle k_T \rangle \sim Q_s$. The assumption is justified since $h_2(k^2 = 0, k'^2 = 0, k \cdot k' = 0, k_T)$ comes from the same amplitude that gave $h_1(k^2 = 0, k'^2 = 0, k \cdot k' = 0, k_T)$, which is equal to the quark spectrum, as shown in Eq. (90), which in turn is always a steeply falling function of k_T scaling at least like $\sim 1/k_T^4$ for k_T above some scale $\langle k_T \rangle \sim Q_s$. By the same argument $h_2(k^2 = 0, k'^2 = 0, k \cdot k' = 0, k_T)$ should be regular (or at most logarithmically divergent) at $k_T = 0$. Using these assumptions we can argue that the integral in the second term on the right hand side of Eq. (91) is dominated by $k_T \sim Q_s$, which allows us to rewrite it as

$$\frac{1}{16\,\pi\, S_\perp}\, h_2(0,0,0,Q_s) \int_0^{Q_s} dk_T\, k_T^3 \left\{ -J_0(k_T\tau)\, J_2(k_T\tau) - 2[J_1(k_T\tau)]^2 + [J_0(k_T\tau)]^2 \right\}. \tag{92}$$

Performing the integration in Eq. (92) one would obtain a linear combination of hypergeometric functions, which can be shown to fall off at least as $\sim 1/\tau^2$ at large τ. Therefore the second term on the right hand side of Eq. (91) falls off with τ at least as $\sim 1/\tau^2$, and can be neglected if the first term on the right hand side of Eq. (91) falls off with τ slower than $\sim 1/\tau^2$.

This can be easily verified. At large τ the first term on the right hand side of Eq. (91) gives

$$\epsilon^{quark}(\tau)\Big|_{\tau \gg 1/\langle k_T \rangle} \approx \frac{2}{\tau} \int d^2k\, \frac{dN^q}{d^2k\, d\eta\, d^2b}\, k_T = \frac{1}{\tau}\, \frac{dE_T^{quarks}}{d\eta\, d^2b}, \tag{93}$$

since the factor of 2 accounts for the anti-quark contribution. We can see that indeed the term in Eq. (92) is negligibly small compared to Eq. (93), which gives us the dominant contribution to the energy density due to quarks at late times τ. (Of course the typical transverse momentum $\langle k_T \rangle$ in Eq. (93) does not have to be exactly equal to the similar typical momentum for gluons in Eq. (53): however, the difference between the two is usually given by the ratio of the Casimir operators, which is just a constant (4/9) and does not change our argument above.)

We have shown that inclusion of massless quarks does not change the conclusion of the previous Sections that the leading diagrammatic contribution to energy density scales as $1/\tau$ at large τ for any order in the coupling g. Therefore, it appears that inclusion of quarks does not affect the onset of thermalization.

6 Conclusions

We have shown above in Eqs. (53) and (93) that gluon and quark fields generated by Feynman diagrams in high energy heavy ion collisions lead to energy densities scaling as $\epsilon \sim 1/\tau$ at

$\tau \gg 1/\langle k_T \rangle$. In the saturation/Color Glass picture of heavy ion collisions, the typical momentum $\langle k_T \rangle$ is proportional to the saturation scale Q_s. Therefore, the $1/\tau$ scaling of energy density sets in at relatively *early* times $\tau \sim 1/Q_s$ (even though throughout the paper we called these proper times *late* times). The remaining evolution of the system in the perturbative scenario considered above, characterized by $\epsilon \sim 1/\tau$ scaling, is reminiscent of the so-called *free streaming*, where the system simply falls apart without particles interacting.

To explain how this happens, let us provide a diagrammatic interpretation of our conclusion of $\epsilon \sim 1/\tau$ scaling. Let us imagine a general gluon field produced in a heavy ion collision as shown in Fig. 2A. There the gluon field is first produced in the nuclear collision at $\tau = 0$ denoted by the $\otimes$ sign. The cross at the other end denotes the later point in τ where we measure the energy density of the gluon field. In the evolution of the system the gluon interacts with other gluon fields produced in the collision in all possible ways, as shown in Fig. 2A. However, the proper times of these interactions are not fixed: they are integrated over the whole range of τ. Interactions may also happen at different impact parameters, which are also integrated over. The $\epsilon \sim 1/\tau$ scaling conclusion from Eqs. (53) and (93) appears to indicate that the dominant diagrams are given by Fig. 2B, where all the interactions happen at early times, after which the system simply falls apart. In other words, the integrations over proper times of the interactions in Fig. 2A are dominated by early times of Fig. 2B. $\langle k_T \rangle$, or Q_s, being the only scale in the problem, sets the typical time scale for the end of interaction period and the onset of free streaming, $\tau_0 \sim 1/Q_s$. Such behavior has been previously observed in the numerical simulations of the classical gluon fields [57, 59].

Another way to physically understand our conclusion of $\epsilon \sim 1/\tau$ scaling is as follows. At any order in the coupling constant α_s the gluon (or quark) field has a diagram (or several diagrams) which is (are) non-zero if the gluon is put on mass-shell. This is just a statement that gluon (or quark) multiplicity distribution can be expanded in a perturbation series in α_s. As we have shown above in Sections 3.3, 4.2 and 5, such diagrams always give energy density scaling as $1/\tau$. Each diagram is dominated by the on-shell particles free streaming away, which always leads to $\epsilon \sim 1/\tau$.

Therefore we have shown that the onset of thermalization and the subsequent Bjorken or rapidity-dependent hydrodynamic expansion of the system of quarks and gluons produced in

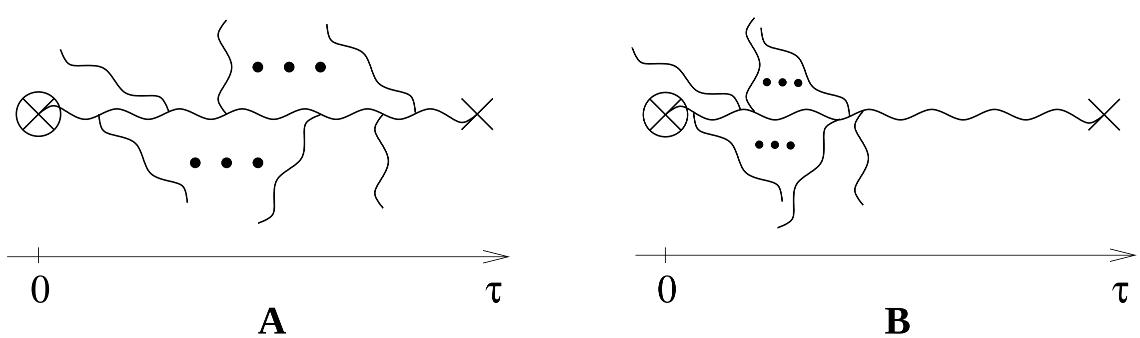

Figure 2: (A) Gluon field produced in a collision with all the interactions throughout its proper time evolution. (B) The dominant contribution to the gluon field comes from early-time interactions.

heavy ion collisions can not result from summation of Feynman diagrams. Nevertheless, there exists a solid phenomenological evidence for the strong final state interactions [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and hydrodynamic behavior [2, 3, 4, 5] of the system produced in heavy ion collisions at RHIC, indicating a formation of strongly interacting quark-gluon plasma (QGP). To reconcile it with the above argument that Feynman diagrams lead to a free streaming behavior for both quarks and gluons, one must conclude that non-perturbative QCD effects are instrumental in QGP formation at RHIC. These could be the the non-perturbative effects associated with infrared modes having momenta of the order of Λ_{QCD} which can not be represented by Feynman diagrams. Therefore, the above argument does not apply to such modes. Alternatively, the non-perturbative effects might be of the nature similar to the ultra-soft modes in finite temperature non-Abelian field theories, which have momenta of the order of $g^2 T$ with T the temperature of the system. It is well-known that resummation of ultra-soft modes is a non-perturbative problem in finite temperature QCD [76]. It is also known that ultra-soft modes are very important for many physical observables for equilibrium QCD matter at finite temperature [77, 78, 79]. If they are important for equilibrium QCD matter, it would be natural to suggest that the ultra-soft modes could also play a major role in non-equilibrium phenomena such as the onset of thermalization. However, a more careful analysis of the issue is needed in order to draw any conclusions. Such analysis is beyond the scope of this paper.

Distinguishing which one of the two types of non-perturbative effects plays a more important role in the process of thermalization would also be important for our understanding of LHC heavy ion data. The non-perturbative effects characterized by the scale Λ_{QCD} are likely to be of little importance at LHC where the saturation scale Q_s is predicted to be much larger than Λ_{QCD} shifting most partons away from the infrared region. At the same time, the non-perturbative ultra-soft modes carrying momenta $g^2 T$ may remain important even at high LHC energies if the relevant temperature scales with the saturation scale, $T \sim Q_s$, increasing at high energy.

Acknowledgments

The author would like to thank Ian Balitsky, Eric Braaten, Ulrich Heinz, Larry McLerran, Al Mueller and Dam Son for many informative discussions. The author is also grateful to Ulrich Heinz for proofreading the manuscript.

This work is supported in part by the U.S. Department of Energy under Grant No. DE-FG02-05ER41377.

Appendix A

Here we are going to prove the following formula

$$I \equiv \int_{-\infty}^{\infty} dk_+ \, dk_- \, e^{-ik_+ x_- - ik_- x_+} \, (k^2 + i\epsilon k_0)^{\Delta - 1} \, = \, -\frac{2\pi^2}{\Gamma(1-\Delta)} \left(\frac{2\, k_T}{\tau} \right)^{\Delta} \, e^{i\,\pi\,\Delta} \, J_{-\Delta}(k_T \tau) \quad \text{(A1)}$$

with $k^2 = 2\,k_+\,k_- - \underline{k}^2$ and for $x_+ > 0$, $x_- > 0$, and $\Delta > 0$. Let us first rewrite the integral (A1) as

$$I \;=\; 2^{\Delta-1} \int_{-\infty}^{\infty} \frac{dk_+\,dk_-\,e^{-ik_+x_- - ik_-x_+}}{(k_+ + i\epsilon)^{1-\Delta}} \left(k_- - \frac{\underline{k}^2}{2(k_+ + i\epsilon)} + i\epsilon \right)^{\Delta-1}. \tag{A2}$$

Defining $\tilde{k}_- = k_- - \underline{k}^2/2k_+$ we write

$$I \;=\; 2^{\Delta-1} \int_{-\infty}^{\infty} \frac{dk_+}{(k_+ + i\epsilon)^{1-\Delta}}\, e^{-ik_+x_- - i\frac{\underline{k}^2}{2k_+ + i\epsilon}\,x_+} \int_{-\infty}^{\infty} d\tilde{k}_-\, e^{-i\tilde{k}_-\,x_+}\,(\tilde{k}_- + i\epsilon)^{\Delta-1}. \tag{A3}$$

The $\tilde{k}_-$ integral can be easily performed by distorting the integration contour around the branch cut. We obtain

$$I \;=\; -2^{\Delta-1}\, \frac{2\pi i\, e^{i\frac{\pi}{2}\Delta}}{\Gamma(1-\Delta)}\, x_+^{-\Delta} \int_{-\infty}^{\infty} \frac{dk_+}{(k_+ + i\epsilon)^{1-\Delta}}\, e^{-ik_+x_- - i\frac{\underline{k}^2}{2k_+ + i\epsilon}\,x_+}. \tag{A4}$$

Expanding the second term in the power of the exponent in Eq. (A4) in a Taylor series we write

$$I \;=\; -2^{\Delta-1}\, \frac{2\pi i\, e^{i\frac{\pi}{2}\Delta}}{\Gamma(1-\Delta)}\, x_+^{-\Delta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\underline{k}^2\,x_+}{2} \right)^n \int_{-\infty}^{\infty} \frac{dk_+}{(k_+ + i\epsilon)^{n+1-\Delta}}\, e^{-ik_+x_-}. \tag{A5}$$

Performing the k_+ integration just like we did the $\tilde{k}_-$ integral above yields

$$I \;=\; -2^{\Delta-1}\, \frac{(2\pi)^2\, e^{i\pi\Delta}}{\Gamma(1-\Delta)}\, (x_+\,x_-)^{-\Delta} \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+1-\Delta)} \left(\frac{-\underline{k}^2\,x_+\,x_-}{2} \right)^n. \tag{A6}$$

Remembering that $2x_+x_- = \tau^2$ and performing the summation over n we obtain Eq. (A1) as desired.

Appendix B

Our goal in this appendix is to perform the following integration

$$J \equiv \int_{-\infty}^{\infty} dk_+\,dk_-\,e^{-ik_+x_- - ik_-x_+}\,(k^2 + i\epsilon k_0)^{\Delta-1}\,(k_+ + i\epsilon)^{\lambda}. \tag{B1}$$

Repeating the steps from Appendix A which led to Eq. (A4) we write

$$J \;=\; -2^{\Delta-1}\, \frac{2\pi i\, e^{i\frac{\pi}{2}\Delta}}{\Gamma(1-\Delta)}\, x_+^{-\Delta} \int_{-\infty}^{\infty} \frac{dk_+}{(k_+ + i\epsilon)^{1-\Delta-\lambda}}\, e^{-ik_+x_- - i\frac{\underline{k}^2}{2k_+ + i\epsilon}\,x_+}. \tag{B2}$$

Expanding the second term in the exponent yields

$$J \;=\; -2^{\Delta-1}\, \frac{2\pi i\, e^{i\frac{\pi}{2}\Delta}}{\Gamma(1-\Delta)}\, x_+^{-\Delta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i\underline{k}^2\,x_+}{2} \right)^n \int_{-\infty}^{\infty} \frac{dk_+}{(k_+ + i\epsilon)^{n+1-\Delta-\lambda}}\, e^{-ik_+x_-}. \tag{B3}$$

Performing the k_+-integration we obtain

$$J \;=\; -2^{\Delta-1}\, \frac{(2\pi)^2\, e^{i\pi\Delta + i\frac{\pi}{2}\lambda}}{\Gamma(1-\Delta)}\, (x_+\,x_-)^{-\Delta}\, x_-^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+1-\Delta-\lambda)} \left(\frac{-\underline{k}^2\,x_+\,x_-}{2} \right)^n, \tag{B4}$$

which, after summing over n gives

$$J = -\frac{2\pi^2}{\Gamma(1-\Delta)} \left(\frac{2\,k_T}{\tau}\right)^{\Delta} \left(\frac{k_T\,\tau}{2\,x_-}\right)^{\lambda} e^{i\,\pi\,\Delta + i\frac{\pi}{2}\lambda}\, J_{-\Delta-\lambda}(k_T\tau). \tag{B5}$$

Noting that $x_\pm = \tau e^{\pm\eta}/\sqrt{2}$ with η the space-time rapidity we rewrite Eq. (B5) as

$$J = -\frac{2\pi^2}{\Gamma(1-\Delta)} \left(\frac{2\,k_T}{\tau}\right)^{\Delta} \left(\frac{k_T}{\sqrt{2}}\right)^{\lambda} e^{\lambda\,\eta}\, e^{i\,\pi\,\Delta + i\frac{\pi}{2}\lambda}\, J_{-\Delta-\lambda}(k_T\tau). \tag{B6}$$

As one can see from Eq. (B6), extra powers of k_+ in Eq. (B1) as opposed to Eq. (A1) do not bring in any extra inverse powers of τ: they only modify the order of the Bessel function.

Finally, let us list here another useful integral, which can be easily obtained by direct integration

$$\int_{-\infty}^{\infty} dk_+\, dk_-\, \frac{e^{-ik_+x_- - ik_-x_+}}{k^2 + i\epsilon k_0}\, k_+^n\, k_-^m = -2\,\pi^2 \left(\frac{i\,k_T\,\tau}{2\,x_-}\right)^n \left(\frac{-i\,k_T\,\tau}{2\,x_+}\right)^m J_{m-n}(k_T\tau), \tag{B7}$$

where n and m are integers.

References

[1] J. D. Bjorken, Phys. Rev. D **27**, 140 (1983).

[2] K. J. Eskola, K. Kajantie and P. V. Ruuskanen, Eur. Phys. J. C **1**, 627 (1998) [arXiv:nucl-th/9705015].

[3] P. F. Kolb, U. W. Heinz, P. Huovinen, K. J. Eskola and K. Tuominen, Nucl. Phys. A **696**, 197 (2001) [arXiv:hep-ph/0103234]; P. Huovinen, P. F. Kolb, U. W. Heinz, P. V. Ruuskanen and S. A. Voloshin, Phys. Lett. B **503**, 58 (2001) [arXiv:hep-ph/0101136]; P. F. Kolb, P. Huovinen, U. W. Heinz and H. Heiselberg, Phys. Lett. B **500**, 232 (2001) [arXiv:hep-ph/0012137]; P. F. Kolb, J. Sollfrank and U. W. Heinz, Phys. Rev. C **62**, 054909 (2000) [arXiv:hep-ph/0006129].

[4] D. Teaney, J. Lauret and E. V. Shuryak, arXiv:nucl-th/0110037; Nucl. Phys. A **698**, 479 (2002) [arXiv:nucl-th/0104041]; Phys. Rev. Lett. **86**, 4783 (2001) [arXiv:nucl-th/0011058]; D. Teaney and E. V. Shuryak, Phys. Rev. Lett. **83**, 4951 (1999) [arXiv:nucl-th/9904006].

[5] T. Hirano and Y. Nara, Nucl. Phys. A **743**, 305 (2004) [arXiv:nucl-th/0404039].

[6] S. S. Adler [PHENIX Collaboration], arXiv:nucl-ex/0306021.

[7] B. B. Back [PHOBOS Collaboration], arXiv:nucl-ex/0306025.

[8] J. Adams [STAR Collaboration], arXiv:nucl-ex/0306024.

[9] I. Arsene *et al.* [BRAHMS Collaboration], Phys. Rev. Lett. **91**, 072305 (2003) [arXiv:nucl-ex/0307003].

[10] K. Adcox *et al.* [PHENIX Collaboration], Phys. Lett. B **561**, 82 (2003) [arXiv:nucl-ex/0207009]; S. S. Adler *et al.* [PHENIX Collaboration], arXiv:nucl-ex/0304022.

[11] B. B. Back *et al.* [PHOBOS Collaboration], arXiv:nucl-ex/0302015.

[12] J. Adams *et al.* [STAR Collaboration], arXiv:nucl-ex/0305015; C. Adler *et al.* [STAR Collaboration], Phys. Rev. Lett. **89**, 202301 (2002) [arXiv:nucl-ex/0206011].

[13] J. D. Bjorken, FERMILAB-PUB-82-059-THY (unpublished).

[14] X. N. Wang, M. Gyulassy and M. Plumer, Phys. Rev. D **51**, 3436 (1995) [arXiv:hep-ph/9408344]; M. Gyulassy, P. Levai and I. Vitev, Phys. Rev. Lett. **85**, 5535 (2000) [arXiv:nucl-th/0005032]; M. Gyulassy, I. Vitev, X. N. Wang and B. W. Zhang, arXiv:nucl-th/0302077; M. Gyulassy, I. Vitev and X. N. Wang, Phys. Rev. Lett. **86**, 2537 (2001) [arXiv:nucl-th/0012092]; I. Vitev and M. Gyulassy, Phys. Rev. Lett. **89**, 252301 (2002) [arXiv:hep-ph/0209161]; X. N. Wang, arXiv:nucl-th/0305010; E. Wang and X.-N. Wang, Phys. Rev. Lett. **89**, 162301 (2002).

[15] R. Baier, Y. L. Dokshitzer, A. H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B **483**, 291 (1997) [arXiv:hep-ph/9607355]; Nucl. Phys. B **484**, 265 (1997) [arXiv:hep-ph/9608322]; Nucl. Phys. B **478**, 577 (1996) [arXiv:hep-ph/9604327]; R. Baier, Y. L. Dokshitzer, A. H. Mueller and D. Schiff, Phys. Rev. C **58**, 1706 (1998) [arXiv:hep-ph/9803473]; Nucl. Phys. B **531**, 403 (1998) [arXiv:hep-ph/9804212]; JHEP **0109**, 033 (2001) [arXiv:hep-ph/0106347]; R. Baier, Y. L. Dokshitzer, S. Peigne and D. Schiff, Phys. Lett. B **345**, 277 (1995) [arXiv:hep-ph/9411409].

[16] A. Kovner and U. A. Wiedemann, arXiv:hep-ph/0304151 and references therein.

[17] B.G. Zakharov, JETP Lett. **63**, 952 (1996); **65**, 615 (1997).

[18] C.A. Salgado and U.A. Wiedemann, Phys. Rev. Lett. **89**, 092303 (2002); Phys. Rev. D **68**, 014008 (2003) [arXiv:hep-ph/0302184].

[19] J. Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).

[20] R. Baier, A.H. Mueller, D. Schiff and D.T. Son, Phys. Lett. **B502**, 51 (2001); Phys. Lett. **B539**, 46 (2002).

[21] A. H. Mueller, Phys. Lett. B **475**, 220 (2000) [arXiv:hep-ph/9909388]; Nucl. Phys. B **572**, 227 (2000) [arXiv:hep-ph/9906322].

[22] J. Serreau and D. Schiff, J. High Energy Phys. **0111**, 039 (2001).

[23] J. Bjoraker and R. Venugopalan, Phys. Rev. C **63**, 024609 (2001).

[24] A. Dumitru and M. Gyulassy, Phys. Lett. **B494**, 215 (2000).

[25] D. Molnar and M. Gyulassy, Nucl. Phys. **A697**, 495 (2002).

[26] S. M. H. Wong, arXiv:hep-ph/0404222.

[27] L. V. Gribov, E. M. Levin and M. G. Ryskin, Phys. Rept. **100**, 1 (1983).

[28] A. H. Mueller and J. w. Qiu, Nucl. Phys. B **268**, 427 (1986).

[29] J. P. Blaizot and A. H. Mueller, Nucl. Phys. B **289**, 847 (1987).

[30] L. D. McLerran and R. Venugopalan, Phys. Rev. D **49**, 2233 (1994) [arXiv:hep-ph/9309289]; Phys. Rev. D **49**, 3352 (1994) [arXiv:hep-ph/9311205]; Phys. Rev. D **50**, 2225 (1994) [arXiv:hep-ph/9402335].

[31] Y. V. Kovchegov, Phys. Rev. D **54**, 5463 (1996) [arXiv:hep-ph/9605446]; Phys. Rev. D **55**, 5445 (1997) [arXiv:hep-ph/9701229].

[32] J. Jalilian-Marian, A. Kovner, L. D. McLerran and H. Weigert, Phys. Rev. D **55**, 5414 (1997) [arXiv:hep-ph/9606337].

[33] A. H. Mueller, Nucl. Phys. B **415**, 373 (1994); A. H. Mueller and B. Patel, Nucl. Phys. B **425**, 471 (1994) [arXiv:hep-ph/9403256]; A. H. Mueller, Nucl. Phys. B **437**, 107 (1995) [arXiv:hep-ph/9408245]; Z. Chen and A. H. Mueller, Nucl. Phys. B **451**, 579 (1995).

[34] I. Balitsky, Nucl. Phys. B **463**, 99 (1996) [arXiv:hep-ph/9509348]; Y. V. Kovchegov, Phys. Rev. D **60**, 034008 (1999) [arXiv:hep-ph/9901281]; Phys. Rev. D **61**, 074018 (2000) [arXiv:hep-ph/9905214].

[35] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, Nucl. Phys. B **504**, 415 (1997) [arXiv:hep-ph/9701284]; Phys. Rev. D **59**, 014014 (1999) [arXiv:hep-ph/9706377]; Phys. Rev. D **59**, 034007 (1999) [Erratum-ibid. D **59**, 099903 (1999)] [arXiv:hep-ph/9807462]; J. Jalilian-Marian, A. Kovner and H. Weigert, Phys. Rev. D **59**, 014015 (1999) [arXiv:hep-ph/9709432]; E. Iancu, A. Leonidov and L. D. McLerran, Phys. Lett. B **510**, 133 (2001) [arXiv:hep-ph/0102009]; E. Iancu and L. D. McLerran, Phys. Lett. B **510**, 145 (2001) [arXiv:hep-ph/0103032]; E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. A **703**, 489 (2002) [arXiv:hep-ph/0109115]; E. Iancu, K. Itakura and L. McLerran, Nucl. Phys. A **708**, 327 (2002) [arXiv:hep-ph/0203137]; for a review see E. Iancu and R. Venugopalan, arXiv:hep-ph/0303204.

[36] M. Gyulassy and L. McLerran, arXiv:nucl-th/0405013.

[37] D. Kharzeev and M. Nardi, Phys. Lett. B **507**, 121 (2001) [arXiv:nucl-th/0012025]; D. Kharzeev, E. Levin and M. Nardi, arXiv:hep-ph/0111315; Nucl. Phys. A **730**, 448 (2004) [Erratum-ibid. A **743**, 329 (2004)] [arXiv:hep-ph/0212316]; D. Kharzeev and E. Levin, Phys. Lett. B **523**, 79 (2001) [arXiv:nucl-th/0108006].

[38] R. Debbe, [BRAHMS Collaboration], talk given at the APS DNP Meeting at Tucson, AZ, October, 2003; R. Debbe [BRAHMS Collaboration], arXiv:nucl-ex/0403052.

[39] I. Arsene *et al.* [BRAHMS Collaboration], arXiv:nucl-ex/0403005.

[40] M. Liu [PHENIX Collaboration], Talk at Quark Matter 2004 Conference, Oakland, California, January 2004; arXiv:nucl-ex/0403047.

[41] P. Steinberg [PHOBOS Collaboration], Talk at Quark Matter 2004 Conference, Oakland, California, January 2004.

[42] L. Barnby [STAR Collaboration], Talk at Quark Matter 2004 Conference, Oakland, California, January 2004; arXiv:nucl-ex/0404027.

[43] D. Kharzeev, E. Levin and L. McLerran, Phys. Lett. B **561**, 93 (2003) [arXiv:hep-ph/0210332].

[44] D. Kharzeev, Y. V. Kovchegov and K. Tuchin, Phys. Rev. D **68**, 094013 (2003) [arXiv:hep-ph/0307037]; Phys. Lett. B **599**, 23 (2004) [arXiv:hep-ph/0405045].

[45] J. L. Albacete, N. Armesto, A. Kovner, C. A. Salgado and U. A. Wiedemann, Phys. Rev. Lett. **92**, 082001 (2004) [arXiv:hep-ph/0307179].

[46] R. Baier, A. Kovner and U. A. Wiedemann, Phys. Rev. D **68**, 054009 (2003) [arXiv:hep-ph/0305265].

[47] J. Jalilian-Marian, Y. Nara and R. Venugopalan, Phys. Lett. B **577**, 54 (2003) [arXiv:nucl-th/0307022].

[48] Yu. V. Kovchegov and A. H. Mueller, Nucl. Phys. B **529**, 451 (1998) [arXiv:hep-ph/9802440].

[49] A. Dumitru and L. D. McLerran, Nucl. Phys. A **700**, 492 (2002) [arXiv:hep-ph/0105268].

[50] Yu. V. Kovchegov and K. Tuchin, Phys. Rev. D **65**, 074026 (2002) [arXiv:hep-ph/0111362].

[51] M. A. Braun, Phys. Lett. B **483**, 105 (2000) [arXiv:hep-ph/0003003].

[52] B. Z. Kopeliovich, A. V. Tarasov and A. Schafer, Phys. Rev. C **59**, 1609 (1999) [arXiv:hep-ph/9808378]; B. Z. Kopeliovich, J. Nemchik, A. Schafer and A. V. Tarasov, Phys. Rev. Lett. **88**, 232303 (2002) [arXiv:hep-ph/0201010].

[53] A. Accardi and M. Gyulassy, arXiv:nucl-th/0308029; X. N. Wang, Phys. Rev. C **61** (2000) 064910; Phys. Lett. B **565** (2003) 116; G. G. Barnafoldi, G. Papp, P. Levai and G. Fai, arXiv:nucl-th/0307062; for a review see M. Gyulassy, I. Vitev, X. N. Wang and B. W. Zhang, arXiv:nucl-th/0302077; B. Z. Kopeliovich, A. V. Tarasov and A. Schafer, Phys. Rev. C **59**, 1609 (1999) [arXiv:hep-ph/9808378].

[54] A. Kovner, L. D. McLerran and H. Weigert, Phys. Rev. D **52**, 3809 (1995) [arXiv:hep-ph/9505320]; Phys. Rev. D **52**, 6231 (1995) [arXiv:hep-ph/9502289].

[55] Y. V. Kovchegov and D. H. Rischke, Phys. Rev. C **56**, 1084 (1997) [arXiv:hep-ph/9704201].

[56] M. Gyulassy and L. D. McLerran, Phys. Rev. C **56**, 2219 (1997) [arXiv:nucl-th/9704034].

[57] A. Krasnitz and R. Venugopalan, Nucl. Phys. B **557**, 237 (1999) [arXiv:hep-ph/9809433]; Phys. Rev. Lett. **84**, 4309 (2000) [arXiv:hep-ph/9909203]; Phys. Rev. Lett. **86**, 1717 (2001) [arXiv:hep-ph/0007108]; A. Krasnitz, Y. Nara and R. Venugopalan, Phys. Rev. Lett. **87**, 192302 (2001) [arXiv:hep-ph/0108092].

[58] Y. V. Kovchegov, Nucl. Phys. A **692**, 557 (2001) [arXiv:hep-ph/0011252]; Nucl. Phys. A **698**, 619 (2002) [arXiv:hep-ph/0106313].

[59] A. Krasnitz, Y. Nara and R. Venugopalan, Nucl. Phys. A **717**, 268 (2003) [arXiv:hep-ph/0209269].

[60] T. Lappi, Phys. Rev. C **67**, 054903 (2003) [arXiv:hep-ph/0303076]; Phys. Rev. C **70**, 054905 (2004) [arXiv:hep-ph/0409328].

[61] A. H. Mueller and D. T. Son, Phys. Lett. B **582**, 279 (2004) [arXiv:hep-ph/0212198]; S. Jeon, arXiv:hep-ph/0412121; S. p. Li and L. D. McLerran, Nucl. Phys. B **214**, 417 (1983); T. Stockamp, arXiv:hep-ph/0408206.

[62] S. M. H. Wong, Nucl. Phys. A **607**, 442 (1996) [arXiv:hep-ph/9606305]; Phys. Rev. C **54**, 2588 (1996) [arXiv:hep-ph/9609287]; Phys. Rev. C **56**, 1075 (1997) [arXiv:hep-ph/9706348].

[63] P. Arnold, J. Lenaghan and G. D. Moore, JHEP **0308**, 002 (2003) [arXiv:hep-ph/0307325].

[64] P. Arnold and J. Lenaghan, Phys. Rev. D **70**, 114007 (2004) [arXiv:hep-ph/0408052].

[65] U. W. Heinz, Nucl. Phys. A **418** (1984) 603C.

[66] S. Mrowczynski, Phys. Lett. B **214**, 587 (1988); **314**, 118 (1993); **393**, 26 (1997) [arXiv:hep-ph/9606442]; Phys. Rev. C **49**, 2191 (1994); Phys. Rev. D **65**, 117501 (2002) [arXiv:hep-ph/0112100]; S. Mrowczynski and M. H. Thoma, Phys. Rev. D **62**, 036011 (2000) [arXiv:hep-ph/0001164].

[67] J. Randrup and S. Mrowczynski, Phys. Rev. C **68**, 034909 (2003) [arXiv:nucl-th/0303021].

[68] P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003) [arXiv:hep-ph/0304092]; arXiv:hep-ph/0406188.

[69] S. G. Matinyan, B. Muller and D. H. Rischke, Phys. Rev. C **57**, 1927 (1998) [arXiv:nucl-th/9708053].

[70] P. Arnold, J. Lenaghan, G. D. Moore and L. G. Yaffe, arXiv:nucl-th/0409068; A. Rebhan, P. Romatschke and M. Strickland, arXiv:hep-ph/0412016.

[71] D. Kharzeev and K. Tuchin, arXiv:hep-ph/0501234.

[72] P. Arnold, G. D. Moore and L. G. Yaffe, JHEP **0011**, 001 (2000) [arXiv:hep-ph/0010177]; JHEP **0305**, 051 (2003) [arXiv:hep-ph/0302165].

[73] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **45**, 199 (1977) [Zh. Eksp. Teor. Fiz. **72**, 377 (1977)]; I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28**, 822 (1978) [Yad. Fiz. **28**, 1597 (1978)].

[74] Y. V. Kovchegov and M. Strikman, Phys. Lett. B **516**, 314 (2001) [arXiv:hep-ph/0107015] and references therein.

[75] I. Balitsky, Phys. Rev. D **70**, 114030 (2004) [arXiv:hep-ph/0409314].

[76] A. D. Linde, Phys. Lett. B **96**, 289 (1980).

[77] D. Bodeker, Phys. Lett. B **426**, 351 (1998) [arXiv:hep-ph/9801430].

[78] P. Arnold, D. T. Son and L. G. Yaffe, Phys. Rev. D **59**, 105020 (1999) [arXiv:hep-ph/9810216].

[79] J. P. Blaizot and E. Iancu, Phys. Rept. **359**, 355 (2002) [arXiv:hep-ph/0101103] and references therein.